DIRECTORATE OF DISTANCE EDUCATION

UNIVERSITY OF NORTH BENGAL

MASTERS OF SCIENCE-MATHEMATICS

SEMESTER -I

ABSTRACT ALGEBRA

DEMATH-1 CORE-1

BLOCK-2

UNIVERSITY OF NORTH BENGAL

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FOREWORD

The Self-Learning Material (SLM) is written with the aim of providing simple and organized study content to all the learners. The SLMs are prepared on the framework of being mutually cohesive, internally consistent and structured as per the university's syllabi. It is a humble attempt to give glimpses of the various approaches and dimensions to the topic of study and to kindle the learner's interest to the subject

We have tried to put together information from various sources into this book that has been written in an engaging style with interesting and relevant examples. It introduces you to the insights of subject concepts and theories and presents them in a way that is easy to understand and comprehend.

We always believe in continuous improvement and would periodically update the content in the very interest of the learners. It may be added that despite enormous efforts and coordination, there is every possibility for some omission or inadequacy in few areas or topics, which would definitely be rectified in future.

We hope you enjoy learning from this book and the experience truly enrich your learning and help you to advance in your career and future endeavours.

COMPLEX ANALYSIS I

BLOCK 1

BLOCK 2

BLOCK II   COMPLEX ANALYSIS-I

Introduction to the Block

In this block we will go through Elementary Functions…. The Exponential Function, Series, Residues And Poles, Applications Of Residues, Mapping By Elementary Functions ….. Linear Transformations, Conformal Mapping, Schwarz-Christoffel Transformation

UNIT-8: ELEMENTARY FUNCTIONS

STRUCTURE

8.0 OBJECTIVES

After studying this unit, you should be able to:

Learn, Understand about Elementary Functions….. The Exponential Function

The Logarithmic Function

Branches And Derivatives Of Logarithms

Some Identities Involving Logarithms

Complex Exponents

Trigonometric Functions

Hyperbolic Functions

Inverse Trigonometric And Hyperbolic Functions

8.1 INTRODUCTION

In this part of the course we will study some basic complex analysis . This is an extremely useful and beautiful part of mathematics and forms the basis of many techniques employed in many branches of mathematic In this section we will study complex functions of a complex variable , Elementary Functions, The Exponential Function, Logarithmic Function, Branches And Derivatives Of Logarithms, Some Identities Involving Logarithms, Complex Exponents, Trigonometric Functions, Hyperbolic Functions, Inverse Trigonometric And Hyperbolic Functions

8.2 ELEMENTARY FUNCTIONS

We consider here various elementary functions studied in calculus and define corresponding functions of a complex variable. To be specific, we define analytic functions of a complex variable z that reduce to the elementary functions in calculus when z=x + i0. We start by defining the complex exponential function and then use it to develop the others.

THE EXPONENTIAL FUNCTION

As anticipated earlier define here the exponential function ez by writing

$e^z = e^x e^{iy} \ (z = x + iy)$ where Euler's formula

$e^{iy} = \cos y + i \sin y$

is used and y is to be taken in radians. We see from this definition that ez reduces to the usual exponential function in calculus when y=0 ; and, following the convention used in calculus, we often write exp z for ez.

Note that since the positive nth root ife of e is assigned to ex when x=1/n (n =2,3,...), expression tells us that the complex exponential function ez is also life when z=1/n (n=2, 3,...). This is an exception to die convention that would ordinarily require us to interpret $e^{1/n}$ as the set of nth roots of e.

According to definition $e^{x}e^{iy}=e^{X+iy}$; and, as already pointed out in the definition is suggested by the additive property

$e^{X1} e^{X2} — e^{X1+X2}$

of e^{X} in calculus. That property's extension,

$e^{X1} e^{X2} — e^{X1+X2}$

of eX in calculus. That property's extension,

$e^{z1} e^{z2} = e^{z1+Z2}$,

to complex analysis is easy to verify. To do this, we write

$z_1 = X1 + iy1$ and z_2 = *2 + iy2-

Then

$e^{z1} e^{z2} = (e^{X1} e^{iy1})(e^{X2} e^{iy2}) = (e^{X1} e^{X2})(e^{iy1} e^{iy2})$.

But x_1 and x_2 are both real, and we know

$e^{iy1} e^{iy2} = e^{i(y1+y2)}$.

$e^{z1} e^{z2} = e^{(X1+X2)} e^{i(y1+y2)}$.

$(X1 + X2) + i(y1 + y2) = (X1 + iy1) + (X2 + iy2) = z_1 + z_2$,

the right-hand side of this last equation becomes e^{z1+z2}. Property is now established.

Observe how property enables us to write $e^{z1-}{}_{Z2}\ e^{z2}=e^{z1}$, or

$e^{z1}/\ e^{z2}=e^{Zl\sim}{}_{Z2}$.

From this and the fact that e0=1, it follows that $1/e^{z}=e^{-z}$.

There are a number of other important properties of ez that are expected. According to Example, for instance,

$d/dz\ e^{z}=e^{z}$

everywhere in the z plane. Note that the differentiability of ez for all z tells us that e^{z} is entire It is also true that

$e^{z}=0$ for any complex number z.

This is evident upon writing definition in the form

$e^{z}=pe^{i\theta}$ where $p=e^{X}$ and $\theta=y$,

which tells us that

$|e^{z}|=e^{x}$ and $\arg(e^{z})=y$ + Inn (n=0, ±1, ±2,...).

Statement then follows from the observation that \ez\ is always positive.

Some properties of ez are, however, not expected. For example, since

$e^{z+2ni}=e^{z}\ e^{2ni}$ and $e^{2ni}=1$,

we find that ez is periodic, with a pure imaginary period of 2ni:

$e^{z+2ni}=e^{z}$.

For another property of ez that ex does not have, we note that while ex is always positive, ez can be negative for instance, that e^{in}=—1. In fact,

$e^{i(2n+1)n}=e^{i2nn+in}=e^{i2nn}e^{in}$=(1)(—1)=-1 (n=0, ±1, ±2,...).

There are, moreover, values of z such that ez is any given nonzero complex number. This is shown in the next section, where the logarithmic function is developed, and is illustrated in the following example.

EXAMPLE. In order to find numbers z=x + iy such that

e^z=1 + i, we write equation as

$e^x e^{iy}=\sqrt{2}e^{in/4}$.

Then, in view of the statment in italics at the beginning of regarding the equality of two nonzero complex numbers in exponential form,

x=ln√2 and y=(2n+1/4)n (n=0, ±1, ±2,...).

Because ln(ex)=x,

Let the function f(z)=u(x,y) + iv(x, y) be analytic in some domain D. State why the functions

U(x, y)=eu(x'y) cos v(x, y), V(x, y)=eu(x'y) sin v(x, y)

are harmonic in D and why V(x,y) is, in fact, a harmonic conjugate of U(x,y).

Establish the identity

(ez)n=enz (n=0, ±1, ±2,...)

in the following way.

Use mathematical induction to show that it is valid when n=0, 1, 2,... .

Verify it for negative integers n by first recalling from Sec. 7 that

zn=(z-1)m (m=—n=1, 2,...)

when z=0 and writing (ez)n=(1 /ez)m. Then use the result in part (a), together with the property 1 /ez=e—z of the exponential function.

8.3 THE LOGARITHMIC FUNCTION

Our motivation for the definition of the logarithmic function is based on solving the equation

e^w=z

for w, where z is any nonzero complex number. To do this, we note that when z and w are written $z = re^{1}$© *(—n < © < n)* and w=u + iv, equation becomes

$e^{u}e^{iv} = re^{1}$©.

According to the statement in italics at the beginning about the equality of two complex numbers expressed in exponential form, this tells us that

e^u=r and v=θ + 2nn

where n is any integer. Since the equation eu=r is the same as u=ln r, it follows that equation is satisfied if and only if w has one of the values

w = *ln*r + i(© + 2nn) (n = *0, ±1, ±2,...).*

Thus, if we write

logz = ln*r* + *i(*© + *2nn) (n* = 0, ±1, ±2,...), equation tells us that $e^{\log z}$ = z (z = 0),

which serves to motivate expression as the *definition* of the (multiple-valued) logarithmic function of a nonzero complex variable $z = re^{1}$©.

EXAMPLE 1. If $z = -1-\sqrt{3i}$, then $r = 2$ and © = —*2n/3*.

Hence

Log $-1-\sqrt{3i}$ =hi2 + i — + 2/7iz^=hi2 + 2^n — ni

(n=0, ±1, ±2,...).

It should be emphasized that it is not true that the left-hand side of equation with the order of the exponential and logarithmic functions reversed reduces to just z. More precisely, since expression can be written

logz=ln |z| + i argz

and since

$|e^z| = e^1$ and $\arg(e^z) = y + 2n\pi$ ($n = 0, \pm1, \pm2,\ldots$)

when $z = x + iy$, we know that

$log(e^z) = ln |e^z| + i\ arg(e^z) = ln(e^x) + i(y + 2n\pi) = (x + iy) + 2n\pi i$

($n = 0, \pm1, \pm2,\ldots$)

That is,

$\log(e^z) = z + 2n\pi i$ ($n = 0, \pm1, \pm2,\ldots$).

The *principal value* of log *z* is the value obtained from equation (2) when $n = 0$ there and is denoted by Log *z*. Thus

Log $z = \ln r + i\Theta$.

Note that Log z is well defined and single-valued when $z = 0$ and that

$\log z = \text{Log} z + 2n\pi i$ ($n = 0, \pm1, \pm2,\ldots$).

It reduces to the usual logarithm in calculus when z is a positive real number $z = r$. To see this, one need only write $z = re^{i0}$, in which case equation becomes Log$z = \ln r$. That is, Log $r = \ln r$.

EXAMPLE. From expression we find that

$\log 1 = \ln 1 + i(0 + 2n\pi) = 2n\pi i$ ($n = 0, \pm1, \pm2,\ldots$). As anticipated, Log 1=0.

Our final example here reminds us that although we were unable to find logarithms of negative real numbers in calculus, we can now do so.

EXAMPLE. Observe that $\log(-1)=\ln 1 + i(\pi + 2n\pi)=(2n + 1)\pi i$ ($n=0, \pm1, \pm2,\ldots$) and that Log $(-1)=\pi i$.

8.4 BRANCHES AND DERIVATIVES OF LOGARITHMS

If $z = re^{6}$ is a nonzero complex number, the argument Q has any one of the values Q = © + $2nn$ (n = 0, ±1, ±2,...), where © = Arg z. Hence the definition

$$\log z = \ln r + i(© + 2nn) \quad (n = 0, \pm1, \pm2, \ldots)$$

of the multiple-valued logarithmic function in Sec. 30 can be written

$$\log z = \ln r + iQ.$$

If we let a denote any real number and restrict the value of Q in expression so that $a < Q < a + 2n$, the function

$$\log z = \ln r + iQ \quad (r > 0,\ a < Q < a + 2n),$$ *with components*

$$u(r,Q) = \ln r \text{ and } v(r,Q) = Q,$$

is single-valued and continuous in the stated domain. Note that if the function were to be defined on the ray θ=a, it would not be continuous there. For if z is a point on that ray, there are points arbitrarily close to z at which the values of v are near a and also points such that the values of v are near a + 2n.

A branch of a multiple-valued function f is any single-valued function F that is analytic in some domain at each point z of which the value F (z) is one of the values of f. The requirement of analyticity, of course, prevents F from taking on a random selection of the values of f. Observe that for each fixed a, the single-valued function is a branch of the multiple-valued function The function

$$\mathrm{Log}\, z = \ln r + i\& \quad (r > 0, — n < \& < n)$$

is called the principal branch.

A branch cut is a portion of a line or curve that is introduced in order to define a branch F of a multiple-valued function f. Points on the branch cut for F are singular points of F, and any point that is common to all branch cuts of f is called a branch point. The origin and the ray Q=a make up the branch cut for the branch of the logarithmic function. The branch cut for the principal branch consists of the origin and the ray &=n. The origin is evidently a branch point for branches of the multiple-valued logarithmic function.

Special care must be taken in using branches of the logarithmic function, especially since expected identities involving logarithms do not always carry over from calculus.

EXAMPLE. When the principal branch is used, one can see that

$\mathrm{Log}(i^3)=3\ \mathrm{Log}\ i$.

We shall derive some identities involving logarithms that do carry over from calculus, sometimes with qualifications as to how they are to be interpreted.

8.5 SOME IDENTITIES INVOLVING LOGARITHMS

If z_1 and z_2 denote any two nonzero complex numbers, it is straightforward to show that

$\log(Z_1Z_2)=\log z_1 + \log z_2$

This statement, involving a multiple-valued function, is to be interpreted in the same way that the statement

$\arg(Z_1Z_2)=\arg z_1 + \arg z_2$

That is if values of two of the three logarithms are specified, then there is a value of the third such that equation holds.

The verification of statement can be based on statement in the following way. Since $|Z_1z_2|=|z_1||z_2|$ and since these moduli are all positive real numbers, we know from experience with logarithms of such numbers in calculus that

$\ln|Z_1Z_2|=\ln|z_1| +\ln|z_2|$

So it follows from this and equation that

$\ln|Z_1Z_2| + i\arg(z_1 z_2)=(\ln|z_1| + i\arg z_1) + (\ln|z_2| + i\arg z_2)$.

Finally, because of the way in which equations and are to be interpreted

Notes

EXAMPLE. To $z_1=z_2=—1$ and recall from log i=2nni and log(—1)=(2n + i)ni,

where n=0, ±i, ±2,.... Noting that Z1Z2=1 and using the values

log(Z1z2)=0 and log z_1=ni, we find that equations is satisfied when the value logz2=ni is chosen. If, on the other hand, the principal values

Logl=0 and Log(—1)=ni

are used,

Log(z1Z2) = 0 and Log z_1 + log z_2 = *2ni*

for the same numbers z_1 and z_2. Thus statement, which is sometimes true when log is replaced by Log, is not always true when principal values are used in all three of its terms

Verification of the statement

We include here two other properties of log z that will be of special interest in f z is a nonzero complex number, that (n=0 ± 1, ±2,...) for any value of log z that is taken. When n=1, this reduces, of course, to relation is readily verified by writing and noting that each side becomes $r^n e^{in\theta}$.

It is also true that when z=0, That is, the term on the right here has n distinct values, and those values are the nth roots of z. To prove this, we write z=r exp(iθ), where θ is the principal value of arg z.

Because exp(i2kn/n) has distinct values only when k=0, 1,... ,n — 1, the right hand side of equation has only n values. That right-hand side is, in fact, an expression for the nth roots of z, and so it can be written z_1/n. This establishes property, which is actually valid when n is a negative integer too.

Exercise :

Show that property also holds when *n* is a negative integer. Do this by writing $z^{i/n} = (z_1^{/m})^{-1}$ $(m = —n)$, where *n* has any one of the negative values *n* = — *i*, —*2*, ... and using the fact that the property is already known to be valid for positive integers.

Let z denote any nonzero complex number, written *z = re*[1]© *(—n <* © *< n)*, and let *n* denote any fixed positive integer *(n = i, 2,...)*. Show that all of the values of $\log(z^{i/n})$ are given by the equation

, , 1 /,,, [1] , , •© + *2(pn+k)jr*

log(z ') = - In *r* + *;*--------- ,

n n

where *p* = 0, ±i, ±2,. . . and *k* = 0, i, *2,. . . ,n* — i. Then, after writing

i *i* © + 2qn

- logz = - lnr + /---------,

n n n

where *q* = 0, ±i, ±2,. . . , show that the set of values of $\log(z^{i/n})$ is the same as the set of values of *(i/n)* logz. Thus show that $\log(z^{i/n})$ = *(i/n)* logz where, corresponding to a value of $\log(z^{i/n})$ taken on the left, the appropriate value of logz is to be selected on the right, and conversely

Suggestion: Use the fact that the remainder upon dividing an integer by a positive integer *n* is always an integer between 0 and *n* — i, inclusive; that is, when a positive integer *n* is specified, any integer *q* can be written *q* = *pn* + *k*, where *p* is an integer and *k* has one of the values *k* = 0, i, *2*,...,*n* — i.

8.6 COMPLEX EXPONENTS

When z=0 and the exponent c is any complex number, the function zc is defined by means of the equation

zc=ec log z,

Notes

where log z denotes the multiple-valued logarithmic function. provides a consistent definition of zc in the sense that it is already known to be valid when c=n (n=0, ±1, ±2,...) and c=1/n (n=±1, ±2,...). Definition is, in fact, suggested by those particular choices of c.

EXAMPLE . Powers of z are, in general, multiple-valued, as illustrated by writing

i^{2i}=exp(-2i log i)

and then

log i=In 1 + i(n/2 + 2nn)=(2n+1/2)ni

This shows that

i^{-2i}=exp[(4n + 1)n]   (n=0, ±1, ±2,...).

Note that these values of i^{-2i} are all real numbers.

Since the exponential function has the property 1/ez=e-z one can

see that

$1/z^{e}$

Exp(-c log z)=z^{-e}

and, in particular, that $1/i^{2i}=i^{-2i}$. According to expression then,

i^{-2i}=exp[(4n + 1)n]   (n =0, ±1,±2,...).

If $z=re^{i\theta}$ and a is any real number, the branch

logz=ln r + id (r > 0, a < θ < a + 2n)

of the logarithmic function is single-valued and analytic in the indicated domain When that branch is used, it follows that the function zc=exp(c logz) is single-valued and analytic in the same domain. The derivative of such a branch of z^{c} is found by first using the chain rule to write

d/dz z^{c}= c exp |(c-l)ogz|

and then recalling the identity z=exp(logz). That yields the result

$d/dz z^c = cz^{c-1}$

The principal value of z^c occurs when log z is replaced by Log z in definition

P.V. $z^c = e^{c\text{Log } z}$.

Equation also serves to define the principal branch of the function zc on the domain |z| > 0, — n < Arg z < n.

EXAMPLE. The principal branch of $z2^{/3}$ can be written

expQLog ^ = expQlnr + = ^exp^ˉ^^.

Thus

P.V. $z^{l\,!/3}$ = -v^ cos ^ + *i*\/*r*^sin

This function is analytic in the domain r > 0, —n<θ<n, as one can see directly

While familiar laws of exponents used in calculus often carry over to complex analysis, there are exceptions when certain numbers are involved.

EXAMPLE. Consider the nonzero complex numbers

z_1=1 + i, z_2=1 — i, and z3=— 1 — i.

When principal values of the powers are taken

$$(z_1 z_2) = 2 = e^{i\text{Log}2} = e^{i(\ln 2 + i0)} = e^{\ln 2}$$

and

T'J= g^!'L∘g^(1+!) _ *y* (In \\f2-\\-*iJT* /4) _ *e*-*jr*/4gr(ln2)/2^ T'2 =
g^!'L∘g^(1-0 _ *y* (In *sfZ*—*iir*/4) _ eJr/4g!(ln2)/2^

Thus

$(Z1Z2)^1$ = z\\z\\,

as might be expected.

Notes

On the other hand, continuing to use principal values, we see that

$(z_2 z_3)^i = (-2)^i = e^{i\,\mathrm{Log}(-2)} = e^{i(\ln 2 + i\pi)} = e^{-\pi} e^{i \ln 2}$

and

= g!,Log(-1_!') = g!'<bl \/2-I'3JT/4) _ e3jr/4g!(ln2)/2^

Hence

$(z_2 z_3) = \backslash e^{\pi/4} e^{i(\ln 2)/2}] [e^{3\pi/} V'^{(\ln 2)/2}] e^{-2\pi}$,

or

$(z_2 z_3)' = z'_2 z_3\ e^{-2\pi}$.

According to definition, the exponential function with base c, where c is any nonzero complex constant, is written

$c^z = e^{z \log c}$.

Note that although ez is, in general, multiple-valued according to definition the usual interpretation of ez occurs when the principal value of the logarithm is taken. This is because the principal value of log e is unity.

When a value of log c is specified, c^z is an entire function of z.

Exercise

Show that if $z = 0$ and a is a real number, then $|z^a| = \exp(a \ln |z|) = |z|^a$, where the principal value of $|z|^a$ is to be taken.

Let $c = a + bi$ be a fixed complex number, where $c = 0, \pm 1, \pm 2, \ldots$, and note that i^c is multiple-valued. What additional restriction must be placed on the constant c so that the values of $|i^c|$ are all the same *Ans.* c is real.

Let c, $c1$, $c2$, and z denote complex numbers, where $z = 0$. Prove that if all of the powers involved are principal values, then

(a) $z^{c1} z^{c2} = z^{c1+c2}$; (b) $\dfrac{z^{c1}}{z^{c2}} = z^{c1-c2}$;

(c) $(z^c)^n = z^{cn}$ $(n = 1, 2, \ldots)$.

Assuming that $f'(z)$ exists, state the formula for the derivative of $c^{f(z)}$.

8.7 TRIGONOMETRIC FUNCTIONS

Euler's formula tells us that

e^{ix} = cos$x$ + i sin$x$ and e^{-ix} = cos x —

i sin$x$ for every real number x. Hence

$e^{ix} - e^{-ix} = 2i\, sinx$ *and* $e^{ix} + e^{-ix} = 2cosx$.

That is,

Sin i = (eix e—ix)/2i

Cos v =(e^x _|_ e—^x)/2

It is, therefore, natural to define the sine and cosine functions of a complex variable z as follows:

eiz - e-iz   eiz + e-iz

It is easy to see from definitions that the sine and cosine functions remain odd and even, respectively:

sin(-z)=- sin z, cos(-z)=cos z.

Also,

eiz=cos z + i sinz.

This is, of course, Euler's formula when z is real.

A variety of identities carry over from trigonometry.

$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2$,

$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$.

From these, it follows readily that

$\sin 2z = 2 \sin z \cos z$, $\cos 2z = \cos^2 z - \sin^2 z$,

Notes

$\sin^2 z + \cos^2 z = 1.$

The periodic character of sin z and cos z is also evident:

sin(z + 2n)=sin z, sin(z + n)=— sin z,

cos(z + 2n)=cos z, cos(z + n)=—cos z.

When y is any real number, definitions and the hyperbolic functions

sin(iy)=i sinhy and cos(iy)=coshy.

Also, the real and imaginary components of sin z and cos z can be displayed in terms of those hyperbolic functions:

sin z=sin x cosh y + i cos x sinh y,

cos z=cos x cosh y — i sin x sinh y, where z=x + iy. To obtain expressions, we write

z_1=x and z_2=iy

derivative of a function

f(z)=u(x, y) + iv(x, y) exists at a point z=(x,y), then

f '(z)=Ux(x, y) + ivx(x, y).

| sinz|2=sin2 x + sinh2 y,

| cosz|2=cos2 x + sinh2 y.

Inasmuch as sinh y tends to infinity as y tends to infinity, it is clear from these two equations that sin z and cos z are not bounded on the complex plane, whereas the absolute values of sin x and cos x are less than or equal to unity for all values of x. A zero of a given function f (z) is a number z0 such that f(z0)=0. Since sin z becomes the usual sine function in calculus when z is real, we know that the real numbers z=nn (n=0, ±1, ±2,...) are all zeros of sinz. To show that there are no other zeros, we assume that sin z=0 and note how it follows from equation that

sin2 x + sinh2 y=0.

This sum of two squares reveals that

sin x=0 and sinh y=0.

Evidently, then, x=nn (n=0, ±1, ±2,...) and y=0; that is,

sinz=0 if and only if z=nn (n=0, ±1, ±2,...).

Since n

cos;=- sin(z)

according to the second of identities

cos;=0 if and only if ;=— + nn (n=0, ±1, ±2,...).

So, as was the case with sin z, the zeros of cos z are all real.

The other four trigonometric functions are defined in terms of the sine and cosine functions by the expected relations:

tanz = sinz/cosz

cotz = cosz/sinz

secz=1/cosz

cscz=1/sinz

Observe that the quotients tan z and sec z are analytic everywhere except at the singularities

Z=— + nn (n=0, ±1, ±2,...),

which are the zeros of cos z. Likewise, cot z and csc z have singularities at the zeros of sin z, namely

z=nn (n=0, ±1, ±2,...).

By differentiating the right-hand sides of equations, we obtain the anticipated differentiation formulas

The periodicity of each of the trigonometric functions defined by equations tan(z + n)=tan z.

Notes

Mapping properties of the transformation w=sin z are especially important in the applications later on. A reader who wishes at this time to learn some of those properties is sufficiently prepared.

8.8 HYPERBOLIC FUNCTIONS

The hyperbolic sine and the hyperbolic cosine of a complex variable are defined as they are with a real variable; that is Because of the way in which the exponential function appears in definitions of sinz and cos z, the hyperbolic sine and cosine functions are closely related to those trigonometric functions:

$-i \sinh(iz) = \sin z, \ \cosh(iz) = \cos z,$

$-i \sin(iz) = \sinh z, \ \cos(iz) = \cosh z.$

Some of the most frequently used identities involving hyperbolic sine and cosine functions are

$\sinh(-z) = -\sinh z, \ \cosh(-z) = \cosh z,$

$\cosh^2 z - \sinh^2 z = 1,$

$\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2,$

$\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$ and

$\sinh z = \sinh x \cos y + i \cosh x \sin y,$

$\cosh z = \cosh x \cos y + i \sinh x \sin y,$

$|\sinh z|^2 = \sinh^2 x + \sin^2 y,$

$|\cosh z|^2 = \sinh^2 x + \cos^2 y,$

where z=x + iy. While these identities follow directly from definitions are often more easily obtained from related trigonometric identities, with the aid of relations.

EXAMPLE. To illustrate the method of proof just suggested, let us verify identity. According to the first of relations, |sinhz|2=|sin(iz)|2. That is,

|sinh z|2=|sin(_y + ix) |2,

where z=x + iy. But from equation we know that

|sin(x + iy)|2=sin2x + sinh2 y ;

and this enables us to write equation in the desired form.

In view of the periodicity of sin z and cos z, it follows immediately from relaions that sinhz and coshz are periodic with period 2ni. Relations together with statements also tell us that

sinhz=0 if and only if z=nni (n=0, ±1, ±2,...) cosh;=0 if and only if z=^— + nn^i (n=0, ±1, ±2,...).

The hyperbolic tangent of z is defined by means of the equation

tanh z =sinh z / cosh z

and is analytic in every domain in which coshz=0. The functions coth z, sech z, and csch z are the reciprocals of tanh z, cosh z, and sinh z, respectively. It is straight-forward to verify the following differentiation formulas, which are the same as those established in calculus for the corresponding functions of a real variable

8.9 INVERSE TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS

Inverses of the trigonometric and hyperbolic functions can be described in terms of logarithms.

In order to define the inverse sine function sin-1 z, we write w=sin-1 z when z=sin w.

That is, w=sin-1 z when

giw g—iw

Z ~ 2i '

If we put this equation in the form

$(e^{iw})^2 — 2iz(e^{iw}) — 1 = 0,$

where $(1 — z^2)^{1/2}$ is, of course, a double-valued function of z. Taking logarithms of each side of equation and recalling that $w=\sin^{-1} z$, we arrive at the expression

$\sin^{-1} z = —i \log[iz + (1 — z^2)^{1/2}]$.

The following example emphasizes the fact that $\sin^{-1} z$ is a multiple-valued function, with infinitely many values at each point z.

EXAMPLE. Expression tells us that

sin–1(— *i)* = —*i* log(l ± *fl).*

But

log(1 + *fl)* =ln(l + *fl)* + 2/7*71 i (n* = 0, ±1, ±2,...)

and

log(l - **V2)** = ln(**V2** - 1) + **(2/7** + 1)**71/ (/7** = 0, ±1, **±2,...).**

Since

ln(**V 2 -** 1) = In ----- - = -ln(l+**V2),**

1+ V 2

then, the numbers

(— 1)" ln(l +*V2) +nni (n* = 0, ±1, ±2,...)

constitute the set of values of log(l ± *fl).* Thus, in rectangular form,

sin–1(— *i) = rnz* + /(—1)"+1 ln(l + *fl) (n* = 0, ±1, ±2,...).

One can apply the technique used to derive expression for sin-1 z to show

that

$\cos^{-1} z = —i \log[z + i(1 - z^2)^{1/2}]$

The functions $\cos^{-1} z$ and $\tan^{-1} z$ are also multiple-valued. When specific branches of the square root and logarithmic functions are used, all three inverse functions become single-valued and analytic because they are then compositions of analytic functions.

The derivatives of these three functions are readily obtained from their logarithmic expressions. The derivatives of the first two depend on the values chosen for the square roots:

— tan z =

dz ~ 1 + z2'

does not, however, depend on the manner in which the function is made single valued.

Inverse hyperbolic functions can be treated in a corresponding manner. It turns out that

$\sinh^{-1} z = \log[z + (z^2 + 1)^{1/2}]$,

$\cosh^{-1} z = \log[z + (z^2 - 1)^{1/2}]$, and Finally, we remark that common alternative notation for all of these inverse functions is arcsin z, etc.

Check your Progress-1

Discuss Elementary Functions

____

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Discuss Complex Exponents

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8.10 LET US SUM UP

Notes

In this unit we have discussed the definition and example of Elementary Functions, The Exponential Function, Logarithmic Function, Branches And Derivatives Of Logarithms, Some Identities Involving Logarithms, Complex Exponents, Trigonometric Functions, Hyperbolic Functions, Inverse Trigonometric And Hyperbolic Functions

8.11 KEYWORDS

Elementary Functions.. In this part of the course we will study some basic complex analysis

The Exponential Function We consider here various elementary functions studied in calculus and define corresponding functions of a complex variable

Logarithmic Function Our motivation for the definition of the logarithmic function is based on solving the equation ew=z

Branches And Derivatives Of Logarithms If z=rel9 is a nonzero complex number, the argument has any one of the values Some Identities Involving Logarithms If z_1 and z_2 denote any two nonzero complex numbers,

Complex Exponents When z=0 and the exponent c is any complex number, the function zc is defined by means of the equation zc=ec log z

Trigonometric Functions Euler's formula tells us that etx=cosx + i sinx and e~tx=cos x — i sinx for every real number x

Hyperbolic Functions The hyperbolic sine and the hyperbolic cosine of a complex variable are defined as they are with a real variable; that is Because of the way in which the exponential function appears in definitions

Inverse Trigonometric And Hyperbolic Functions Inverses of the trigonometric and hyperbolic functions can be described in terms of logarithms. In order to define the inverse sine function sin-1 z, we write w=sin-1 z when z=sin w.

8.12 QUESTIONS FOR REVIEW

Explain Elementary Functions

Explain Complex Exponents

8.13 ANSWERS TO CHECK YOUR PROGRESS

Elementary Functions (answer for Check your Progress-1 Q)

Complex Exponents (answer for Check your Progress-1 Q)

8.14 REFERENCES

- Complex Analysis
- Basic of Complex Analysis
- Complex Functions & Variables
- Introduction To Complex Analysis
- Application Of Complex Analysis & Variables

UNIT-9: SERIES

STRUCTURE

9.0 OBJECTIVES

After studying this unit, you should be able to:

Learn, Understand about Series

Convergence Of Sequence

Convergence Of Series

Taylor Series

Laurent Series

Absolute And Uniform Convergence Of Power Series

Continuity Of Sums Of Power Series

Integration And Differentiation Of Power Series

Uniqueness Of Series Representations

Multiplication And Division Of Power Series

9.1 INTRODUCTION

In this part of the course we will study some basic complex analysis . This is an extremely useful and beautiful part of mathematics and forms the basis of many techniques employed in many branches of mathematic In this section we will study complex functions of a complex variable, Series, Convergence Of Sequence, Convergence Of Series, Taylor Series, Laurent Series, Absolute And Uniform Convergence Of Power Series, Continuity Of Sums Of Power Series, Integration And Differentiation Of Power Series, Uniqueness Of Series Representations, Multiplication And Division Of Power Series

9.2 SERIES

This chapter is devoted mainly to series representations of analytic functions. We present theorems that guarantee the existence of such representations, and we develop some facility in manipulating series.

9.3 CONVERGENCE OF SEQUENCES

An infinite sequence

z_1 ,z_2,---,Zn,...

of complex numbers has a limit z if, for each positive number e, there exists a positive integer n0 such that

|zn — z| < e whenever n > n0.

Geometrically, this means that for sufficiently large values of n, the points zn lie in any given e neighborhood of z. Since we can choose e as small as we please,

it follows that the points zn become arbitrarily close to z as their subscripts increase. Note that the value of no that is needed will, in general, depend on the value of e.

The sequence can have at most one limit. That is, a limit z is unique if it exists. When that limit exists, the sequence is said to converge to z ; and we write

lim zn=z.

If the sequence has no limit, it diverges.

Theorem. Suppose that zn=xn + iyn (n=1, 2,...) andz=x + iy. Then

lim zn=z

if and only if

lim xn=x and lim yn=y.

To prove this theorem, we first assume that conditions hold and obtain condition from it. According to conditions, there exist, for each positive number e, positive integers n\ and n2 such that

\xn — x\ < - whenever n > n \

and

I.Vn — v| < ^ whenever n > «2-

Hence if no is the larger of the two integers n1 and n2,

\xn — x\ < - and |y,, — y| < - whenever n > no.

Since

\(Xn + iyn) - (x + iy)\=\(xn - x) + i(yn - y)| < \xn - x| + \yn - y|,

then,

|zn — z| c — + —=e whenever n > no-

Condition thus holds.

Conversely, if we start with condition, we know that for each positive number e, there exists a positive integer no such that

\ (xn + iyn) - (x + iy)\< e whenever n > n0.

But

\Xn - X\<\(xn - x) + i(yn - y)|=\(xn + iyn) - (x + iy)\

and

\yn - y\<\(xn - x) + i(yn - y)\=\(xn + iyn) - (x + iy)\;

and this means that

\xn - x\ < e and \yn - y\ < e whenever n > no.

That is, conditions are satisfied.

Note how the theorem enables us to write

lim (xn + iyn)=lim xn + i lim yn

n^^ n^^ n^^

whenever we know that both limits on the right exist or that the one on the left exists.

EXAMPLE . The sequence z=\+i (n=1,2)

converges to i since

Notes

lim (—r + M=lim —t + i lim 1 =0 + /l=i.

Definition can also be used to obtain this result. More precisely, for each positive number e,

\zn — 1|=— < e whenever n > -r

One must be careful when adapting our theorem to polar coordinates, as the following example shows.

EXAMPLE . If, using polar coordinates, we write

rn=\zn\ and &n=ArgZn (n=1, 2,...),

where Arg zn denotes principal arguments (—n < & < n) of zn, we find that

lim rn=lim J A + \=2

but that

lim &2n=n and lim &2n—i=— n (n=1, 2,...).

Evidently, then, the limit of &n does not exist as n tends to infinity.

9.4 CONVERGENCE OF SERIES

An infinite series

TO

zn= z_1 + z_2 + …. + zn +...

of complex numbers converges to the sum S if the sequence

Sn=^2 zn= z_1 + z_2 + +zn (N=1, 2,...)

of partial sums converges to S; we then write

zn=S.

Note that since a sequence can have at most one limit, a series can have at most one sum. When a series does not converge, we say that it diverges.

Theorem. Suppose that zn=xn + iyn (n=1, 2,...) and S=X + iY

This theorem can be useful in showing that a number of familiar properties of series in calculus carry over to series whose terms are complex numbers. To illustrate how this is done, we include here two such properties and present them as corollaries.

Corollary. If a series of complex numbers converges, the nth term converges to zero as n tends to infinity.

Assuming that series converges, we know from the theorem that if

Zn=Xn + iyn (n=1, 2,.. .),

then each of the series

OO

xn and E yn

n l   n l

converges. We know, moreover, from calculus that the nth term of a convergent series of real numbers approaches zero as n tends to infinity.

lim zn=lim xn + i lim yn=0 + 0 * i=0;

It follows from this corollary that the terms of convergent series are bounded. That is, when series converges, there exists a positive constant M such that |zn| < M for each positive integer n.

For another important property of series of complex numbers that follows from a corresponding property in calculus series is said to be absolutely convergent if the series

XI yxl + yn (Zn=Xn + iyn)

of real numbers y x^ + y% converges.

Corollary. The absolute convergence of a series of complex number's implies the convergence of that series.

I < y.v-2 + yl and | yn | < yjxl + y*, we know from the comparison test in calculus that the two series must onverge. Moreover, since the absolute convergence of a series of real numbers implies the convergence of the series itself, it follows that the series both converge. In view of the theorem in this section, then, series converges.

In establishing the fact that the sum of a series is a given number S, it is often convenient to define the remainder pN after N terms, using the partial sums :

pn=S - sn-

Thus S=Sn + Pn; and, since |Sn — S|=|pn — 0|, we see that a series converges to a number S if and only if the sequence of remainders tends to zero. We shall make considerable use of this observation in our treatment of power series. They are series of the form

y]an(z — zo)n=ao + a\(z — zo) + a2(z — zo)^2 + ... + an(z — zo)n + ...,

where z0 and the coefficients an are complex constants and z may be any point in stated region containing z0. In such series, involving a variable z, we shall denote sums, partial sums, and remainders by S(z), SN(z), and pN(z), respectively.

It is clear from this that the remainders pN(z) tend to zero when |z| < 1 but not when |z| > 1. Summation formula is, therefore, established.

limits of sequences to verify the limit of the sequence zn (n=1, 2,...)

Let ®n (n=1, 2,...) denote the principal arguments of the numbers

(-1)n

Z,,=2 + 1—(«=1,2).

and compare

Use the inequality ||zn| — |z|| < |zn — z| to show that

if lim Zn=z, then lim |znl=|z|.

when 0 < r < 1. (Note that these formulas are also valid when r=0.)

Show that a limit of a convergent sequence of complex numbers is unique by appealing to the corresponding result for a sequence of real numbers.

9.5 TAYLOR SERIES

We turn now to Taylor's theorem, which is one of the most important results.

Theorem. Suppose that a function f is analytic throughout a disk |z — zo| < Ro, centered at zo and with radius Ro. Then f(z) has the power series representation

f(z) =^2, an(z — Z_0)n ($\left|Z - Z_0\right|$ < Ro),

That is, series converges to f(z) when z lies in the stated open disk.

This is the expansion of f(z) into a Taylor series about the point zo. It is the familiar Taylor series from calculus, adapted to functions of a complex variable. With the agreement that

f(*H.zo)=f(zo) and o!=1,

Any function which is analytic at a point zo must have a Taylor series about zo. For, if f is analytic at z0, it is analytic throughout some neighborhood $\left|Z - Z_0\right|$ < £ of that point and £ may serve as the value of R0 in the statement of Taylor's theorem. Also, if f is entire, R0 can be chosen arbitrarily large; and the condition of validity becomes $\left|Z - Z_0\right|$ < rc>. The series then converges to f(z) at each point z in the finite plane.

When it is known that f is analytic everywhere inside a circle centered at z0, convergence of its Taylor series about z0 to f (z) for each point z within that circle is ensured; no test for the convergence of the series is even required. In fact, according to Taylor's theorem, the series converges to f (z) within the circle about z0 whose radius is the distance from z0 to the nearest point z_1 at which f fails to be analytic. We shall find that this is actually the largest circle centered at z0 such that the series converges to f (z) for all z interior to it.

In the following section, we shall first prove Taylor's theorem when z0=0, in which case f is assumed to be analytic throughout a disk | z| < R0 and series becomes a Maclaurin series:

The proof when z0 is arbitrary will follow as an immediate consequence. A reader who wishes to accept the proof of Taylor's theorem can easily

PROOF OF TAYLOR'S THEOREM

To begin the derivation of representation we write |z|= r and let C0 denote and positively oriented circle |z|=r0, where r < r0 < R0 Since f is analytic inside and on the circle Q and since the point z is interior to Co, the Cauchy integral formula Multiplying through this equation by f(s) and then integrating each side with respect to s around Representation

To verify the theorem when the disk of radius R0 is centered at an arbitrary point Z0, we suppose that f is analytic when $|Z - Z_0|$ < R0 and note that the composite function f(z + z0) must be analytic when \(z + z0) - z0\< R0. This last inequality is, of course, just \z\ < R0 ; and, if we write g(z)=f(z + z0), the analyticity of g in the disk |z| < R0 ensures the existence of a Maclaurin series representation:

After replacing z by z - z0 in this equation and its condition of validity

EXAMPLE . Since the function f(z)=ez is entire, it has a Maclaurin series representation which is valid for all z. Here f(n)(z)=ez (n=0, 1, 2,...);

and, because f(n)(0)=1 (n=0, 1, 2

The entire function $_{Z2}$ e3z also has a Maclaurin series expansion. The simplest way to obtain it is to replace z by 3z on each side of equation and then multiply through the resulting equation by $_{Z2}$:

9.6 LAURENT SERIES

If a function f fails to be analytic at a point z0, one cannot apply Taylor's theorem at that point. It is often possible, however, to find a series representation for f(z) involving both positive and negative powers of z — z0 Laurent's theorem.

Theorem. Suppose that a function f is analytic throughout an annular domain R.1 < $|Z - Z_0|$ < R-l , centered at z0 , and let C denote any positively oriented simple closed contour around z0 and lying in that domain. Then, at each point in the domain, f(z) has the series representation when n > 0,In either one of the forms, the representation of f(z) is called a Laurent series.

Observe that the integrand in expression can be written f(z)(z — Z0)n 1. Thus it is clear that when f is actually analytic throughout the disk $|Z - Z_0|$ < R2, this integrand is too. Hence all of the coefficients bn are zero; and, because reduces to a Taylor series about z0.

If, however, f fails to be analytic at z0 but is otherwise analytic in the disk $|Z - Z_0|$ < R2, the radius R1 can be chosen arbitrarily small. Representation is then valid in the punctured disk 0 < $|Z - Z_0|$ < R2. Similarly, if f is analytic at each point in the finite plane exterior to the circle |z — z01=R1, the condition of validity is R1 < $|Z - Z_0|$ < to. Note that if f is analytic everywhere in the finite plane except at z0, series is valid at each point of analyticity, or when 0 < $|Z - Z_0|$ < ^>.

We shall prove Laurent's theorem first when z0=0, which means that the annulus is centered at the origin. The verification of the theorem when z0 is arbitrary will follow readily; and, as was the case with Taylor's theorem, a reader can skip the entire proof without difficulty.

PROOF OF LAURENT'S THEOREM

We start the proof by forming a closed annula r region ry < |z| < r2 that is contained in the domain R1 < |z| < R2 and whose interior contains both the point z and the contour C We let C1 and C2 denote the circles |z|=n and |z| =r2,

respectively, and we assign them a positive orientation. Observe that f is analytic on Ci and C2, as well as in the annular domain between them.

Next, we construct a positively oriented circle y with center at z and small enough to be contained in the interior of the annular region ri < |z| < ri, It then follows from the adaptation of the Cauchy-Goursat theorem to integrals of analytic functions around oriented boundaries of multiply connected domains that according to the Cauchy integral formula, the value of the third integral here is 2nif(z). Hence

Now the factor 1/(s — z) in the first of these integrals is the same as in expression, where Taylor's theorem was proved; and we shall need here the expansion which was used in that earlier section. As for the factor 1/(z — s) in the second integral, an interchange of s and z in equation reveals.

If we replace the index of summation n here by n — 1, this expansion takes the form which is to be used in what follows.

by f(s)/(2ni) and then integrating each side of the resulting equations with respect to s around C2 and C1, respectively, we find from expression that

where the numbers an (n=0, 1, 2,..., N — 1) and bn (n=1, 2,...,N) are given by the equations f(s) ds

As N tends to <xi, expression evidently takes the proper form of a Laurent series in the domain R1 < |z| < R2, provided that

lim pN(z)=0 and lim oN(z)=0.

These limits are readily established by a method already used in the proof of Taylor's. We write |z|=r, so that r1 < r < ri, and let M denote the maximum value of |f (s)| on C1 and C2. We also note that if s is a point on C2, then |s — z| > r2 — r ; and if s is on C1, we have |z — s| > r — ru

Since (r/r2) < 1 and (r1/r) < 1, it is now clear that both pN(z) and aN(z) tend to zero as N tends to infinity.

Finally the contours used in integrals here may be replaced by the contour C. This completes the proof of Laurent's theorem when zo=0 since, if z is used instead of s as the variable of integration, expressions for the coefficients an and bn are the same when z0=0 there.

To extend the proof to the general case in which z0 is an arbitrary point in the finite plane, we let f be a function satisfying the conditions in the theorem; and, just as we did in the proof of Taylor's theorem, we write g(z)=f(z + z0). Since f(z) is analytic in the annulus R1 < $|Z - Z_0|$ < R2, the function f(z + z0) is analytic when R1 < $|(Z + Z_0) - Z_0|$ < R2. That is, g is analytic in the annulus R1 < |z| < R2, which is centered at the origin. Now the simple closed contour C in the statement of the theorem has some parametric representation z=z(t) (a < t < b), where

$$R1 < |U(t) — z0| < R2$$

for all t in the interval a < t < b. Hence if Y denotes the path

z=z(t) — z0 (a < t < b),

T is not only a simple closed contour but, in view of inequalities it lies in the domain R1 < jzj < R2. Consequently, g(z) has a Laurent series representation

Representation is obtained if we write f(z + z0) instead of g(z) in equation and then replace z by z — z0 in the resulting equation, as well as in the condition of validity R1 < j zj < R2. Expression for the coefficients an is moreover, the same as expression, since

f(z) dz

Similarly, the coefficients bn in expression are the same as those in expression.

EXAMPLE . The function f(z)=1/(z — i)2 is already in the form of a Laurent series, where z0=i. That is all of the other coefficients are zero.

for the coefficients in a Laurent series, we know that

in=0. ±1. ±2 ,

Notes

where C is, for instance, any positively oriented circle |z — i |=R about the point z0=i.

f(z) = (z — 1)(z — 2) z — 1 z — 2 which has the two singular points z=1 and z=2, is analytic in the domains

|z| < 1, 1 < |z| < 2, and 2 < |z| < o.

In each of those domains, denoted by D1, D2, and D3, respectively f(z) has series representations in powers of z. They can all be found by making the appropriate replacements for z in the expansion

EXAMPLE . The representation in Di is a Maclaurin series. To find it, we observe that |z| < 1 and |z/2| < 1 when z is in D1

The representations in D2 and D3 are treated in the next two examples.

EXAMPLE . Because 1 < |z| < 2 when z is a point in D2, we know that |1/z| < 1 and |z/2| < 1 for such points. This suggests writing expression as

If we replace the index of summation n in the first of these series by n — 1 and then interchange the two series, we arrive at an expansion having the same form as the one in the statement of Laurent's theorem Since there is only one Laurent series for f(z) in the annulus D2, expansion is, in fact, the Laurent series for f(z) there.

EXAMPLE . The representation of the function in the unbounded domain D3, where 2 < |z| < 0, is also a Laurent series. Since |2/z| < 1 when z is in D3, it is also true that |1/z| < 1 used in Laurent's theorem in that theorem.

EXERCISES

Find the Laurent series that represents the function

f(z) =z2 sinf in the domain 0 < |z| < 0.

Find a representation for the function

fiz)=1 1 1

1 + z z 1 + (1/z) in negative powers of z that is valid when 1 < |z| < w.

and specify the regions in which those expansions are valid.

Let a denote a real number, where -1 < a < 1, and derive the Laurent series representation.

After writing z=e1 in the equation obtained in part, equate real parts and then imaginary parts on each side of the result to derive the summation formulas converges to an analytic function X(z) in some annulus R1 < |z| < R2. That sum X(z) is called the z-transform of x[n] (n=0, ±1, ±2,.. .). for the coefficients in a Laurent series to show that if the annulus contains the unit circle |z|=1, then the inverse z-transform of X(z)

Let f(z) denote a function which is analytic in some annular domain about the origin that includes the unit circle z=e'^ (—n < <p < n). By taking that circle as the path of integration in expressions coefficients an and bn in a Laurent series in powers of z, show that

1 Cn   1 Cn

when z is any point in the annular domain.

Write u(0)=Re[f(e'θ)] and show how it follows from the expansion This is one form of the Fourier series expansion of the real-valued function u(0) on the interval — n < θ < n. The restriction on u(θ) is more severe than is necessary in order for it to be represented by a Fourier series.

9.7 ABSOLUTE AND UNIFORM CONVERGENCE OF POWER SERIES

This section and the three following it are devoted mainly to various properties of power series. A reader who wishes to simply accept the theorems and the corollary in these sections can easily skip the proofs in order to reach Sec. 67 more quickly.

Notes

We recall from a series of complex numbers converges absolutely if the series of absolute values of those numbers converges. The following theorem concerns the absolute convergence of power series.

Theorem . If a power series

an(z — Z0)n

n= 0

converges when z=$_{Z1}$ (zt=Z0), then it is absolutely convergent at each point z in the open disk |z — zo| < R1 where R1=|$_{Z1}$ — zo|

y

We start the proof by assuming that the series

^ an($_{Z1}$ — zo)n ($_{Z1}$=zo)

converges. The terms an($_{Z1}$ — zo)n are thus bounded; that is,

|an($_{Z1}$ — zo)^{n}\<M (n=0, 1, 2,...) for some positive constant M. If |z — zo| < Ri and if we write

|z — zo|

|$_{Z1}$ — zo|

we can see that

Ia„(z - zo)"I=\a„($_{Z1}$ - zo)"l (….)^{n} < Mp" (n=0, 1, 2,...).

Now the series

J^Mp"

is a geometric series, which converges since p < i. Hence, by the comparison test for series of real numbers,

^an^ — zo)n|

converges in the open disk |z — zo| < Ri. This completes the proof.

The theorem tells us that the set of all points inside some circle centered at zo is a region of convergence for the power series (i), provided it converges at some point other than zo. The greatest circle centered at zo such that series (i) converges at each point inside is called the circle of convergence of series (i). The series cannot converge at any point z_2 outside that circle, according to the theorem ; for if it did, it would converge everywhere inside the circle centered at zo and passing through z_2. The first circle could not, then, be the circle of convergence.

Our next theorem involves terminology that we must first define. Suppose that the power series has circle of convergence |z — zo I=R, and let S(z) and SN(z) represent the sum and partial sums, respectively, of that series:

$S(z)=\hat{}2an(z — zo)^n$, $Sn(z)=\hat{}2\ an(z — zo)^n$ (|z — zo| < R).

Then write the remainder function

PN(z)=S(z) — Sn(z) (|z — zo| < R).

Since the power series converges for any fixed value of z when \z — zo | < R, we know that the remainder pN(z) approaches zero for any such z as N tends to infinity. According to definition of the limit of a sequence, this means that corresponding to each positive number e, there is a positive integer Ne such that

\pn(z)\ < e whenever N > Ne.

When the choice of Ne depends only on the value of e and is independent of the point z taken in a specified region within the circle of convergence, the convergence is said to be uniform in that region.

Theorem. If z_1 is a point inside the circle of convergence \z — z0 \=R of a power series then that series must be uniformly convergent in the closed disk\z — zo\< Ri, where

Ri=\z_1 — zo\

Our proof of this theorem depends on theorem. Given that z_1 is a point lying inside the circle of convergence of series, we note that there are points inside that circle and farther from z0 than z_1 for which the series converges. So, according to converges. Letting m and N denote positive integers.

Since on are the remainders of a convergent series, they tend to zero as N tends to infinity. That is, for each positive number e, an integer Ns exists such that

on < e whenever N > Ne

Because of conditions, then, condition holds for all points z in the disk \z - Z0\< R1; and the value of Ne is independent of the choice of z. Hence the convergence of series is uniform in that disk.

9.8 CONTINUITY OF SUMS OF POWER 1SERIES

Our next theorem is an important consequence of uniform convergence, discussed in the previous section.

Theorem. A power series $\hat{}2an(z - Z0)^n$

represents a continuous function S(z) at each point inside its circle of convergence \z - Z0 \=R

Another way to state this theorem is to say that if S(z) denotes the sum of series within its circle of convergence $|Z - Z_0|$ =R and if z_1 is a point inside that circle, then for each positive number e there is a positive number S such that

\S(z) - S(z 1)\ < e whenever \z - z_1\< S.

To prove the theorem, we let Sn(z) denote the sum of the first N terms of series and write the remainder function

Pn(z)=S(z) — SN(z) ($|Z - Z_0|$ < R)

Then, because

S(z)=Sn(z) + pn(z) ($|Z - Z_0|$ < R),

one can see that

\S(z) — S(z_1)\=\Sn(z) — Sn(z_1) + pn(z) — pn(z_1)\, or

\S(z) — S(z_1)\ < \Sn(z) — Sn(z_1)\ + \pn(z)\ + \pn(z_1)\.

If z is any point lying in some closed disk \z — z01 < R0 whose radius R0 is greater than |z_1 — z0 | but less than the radius R of the circle of convergence of series the uniform convergence stated ensures that there is a positive integer Ns such that

\pn(z)\ < - whenever N > Ns.

In particular, condition holds for each point z in some neighborhood \z — z_1\< S of z_1 that is small enough to be contained in the disk \z — z01 < R0.

Now the partial sum SN(z) is a polynomial and is, therefore, continuous at z_1 for each value of N. In particular, when N=Ns + 1, we can choose our S so small that

|SN(z) -SN(z_1)\ < - whenever \z - z_1\ < 5.

By writing N=Ns + 1 in inequality and using the fact that statements and

are true when N=Ns + 1, we now find that

|5(z) - S(z_1)\ < - + - + - whenever |z — z_1| < 5.

This is statement, and the theorem is now established.

By writing w=1/(z — Z0), one can modify the two theorems in the previous section and the theorem here so as to apply to series of the type

If, for instance, series converges at a point z_1 (z_1=Z0), the series

ybn

n=1

must converge absolutely to a continuous function when

Thus, since inequality is the same as $\left|Z - Z_0\right| > \left|Z_1 - Z_0\right|$, series must converge absolutely to a continuous function in the domain exterior to the circle $|Z - Z_0|$=^1, where R1= $|Z - Z_0|$. Also, we know that if a Laurent series representation

is valid in an annulus R1 < $|Z - Z_0|$ < R2, then both of the series on the right converge uniformly in any closed annulus which is concentric to and interior to that region of validity.

9.9 INTEGRATION AND DIFFERENTIATION OF POWER SERIES

We have just seen that a power series

S(z) =y an(z — Z0)n

represents a continuous function at each point interior to its circle of convergence. In this section, we prove that the sum S(z) is actually analytic within that circle. Our proof depends on the following theorem, which is of interest in itself.

Theorem. Let C denote any contour interior to the circle of convergence of the power series, and let g(z) be any function that is continuous on C. The series formed by multiplying each term of the power series by g(z) can be integrated term by term over C; that is,

f g(z)S(z) dz= n f g(z)(z - zo)^n dz.

To prove this theorem, we note that since both g(z) and the sum S(z) of the power series are continuous on C, the integral over C of the product

g(z)S(z)=^2 ang(z)(z - z0)n + g(z)pN(z),

where pN(z) is the remainder of the given series after N terms, exists. The terms of the finite sum here are also continuous on the contour C, and so their integrals over C exist. Consequently, the integral of the quantity g(z)pN(z) must exist; and we may write

[g(z)S(z) dz=Y] an f g(z)(z - z0)n dz + f g(z)pN(z) dz.

Now let M be the maximum value of |g(z) | on C, and let L denote the length of C. In view of the uniform convergence of the given power series know that for each positive number e there exists a positive integer Ne such that, for all points z on C,

|pN(z)| < e whenever N > Ne.

Since Ne is independent of z, we find that

< MeL whenever N >

that is,

lim/g(z)PN(z) dz=0.

It follows, therefore, from equation that

g(z)S(z) dz=lim an I g(z)(z - z0)n dz.

/ g(z)S(z) dz=lim Y] an

This is the same as equation.If g(z)=1 for each value of z in the open disk bounded by the circle of convergence of power series, the fact that (z — Z0)n is entire when n=0, 1, 2,... ensures that

f g(z)(z — Z0)n dz=f (z — Z0)n dz=0 (n=0, 1, 2,...)

for every closed contour C lying in that domain. According to equation then,

S(z) dz=0

for every such contour; and, by Morera's theorem the function S(z) is analytic throughout the domain. We state this result as a corollary.

Corollary. The sum S(z) of power series is analytic at each point z interior to the circle of convergence of that series.

This corollary is often helpful in establishing the analyticity of functions and in evaluating limits.

EXAMPLE . To illustrate, let us show that the function defined by means of the equations

Notes

f( ) _ j(ez — 1)/z when z=0,   when z=0

is entire. Since the Maclaurin series expansion

represents ez — 1 for every value of z, the representation

obtained by dividing each side of equation by z, is valid when z=0. But series clearly converges to f(0) when z=0. Hence representation is valid for all z; and f is, therefore, an entire function. Note that since (ez — 1)/z=f(z) when z=0 and since f is continuous at z=0,

The first limit here is, of course, also evident if we write it in the form

(ez — I) — 0

z — 0

which is the definition of the derivative of ez — 1 at z=0.

that the Taylor series for a function f about a point Z0 converges to f(z) at each point z interior to the circle centered at Z0 and passing through the nearest point z_1 where f fails to be analytic. In view of our corollary we now know that there is no larger circle about z0 such that at each point z interior to it the Taylor series converges to f(z). For if there were such a circle, f would be analytic at z_1; but f is not analytic at z_1.

Theorem. The power series can be differentiated term by term. That is, at each point z interior to the circle of convergence of that series,

S '(z)=^ na_n(z — z0)n 1

To prove this, let z denote any point interior to the circle of convergence of series. Then let C be some positively oriented simple closed contour surrounding z and interior to that circle

2ni $(s — z)^2$

at each point s on C. Since g(s) is continuous on C, Theorem tells us that

f g(s)S(s) ds=an f g(s)$(s — z0)^n$ ds.

Now S(z) is analytic inside and on C, and this enables us to write

f1 f S(s) ds .

with the aid of the integral representation for derivatives.

9.10 UNIQUENESS OF SERIES REPRESENTATIONS

The uniqueness of Taylor and Laurent series representations, anticipated respectively, follows readily from. We consider first the uniqueness of Taylor series representations.

Theorem . If a series

^2an(z - Z0)n

converges to f(z) at all points interior to some circle $|Z — Z0| = R$, then it is the Taylor series expansion for f in powers of z — Z0 •

To start the proof, we write the series representation

f(z)=Y^ o-n(z — z0)n ($|Z — Z0|$ < R)

in the hypothesis of the theorem using the index of summation m:

f(z)=^ Om(z — z0)m$|Z — Z0|$ (< R).

Then, by appealing we may write

f g(z)f(z) dz=V am f g(z)(z — z0)m dz,

where g(z) is any one of the functions

and C is some circle centered at zo and with radius less than R.

In view of the extension of the Cauchy integral formula we find that

f(z) dz f (zo)

and, since

[, w \m j 1 [dz when m=n,

Notes

it is clear that

^2 am J g(z)(z - Z0)m dz=an.

Because of equations equation now reduces to

f(n)(z0)

This shows that series is, in fact, the Taylor series for f about the point z0.

Note how it follows from that if series converges to zero through- out some neighborhood of z0, then the coefficients an must all be zero.

Our second theorem here concerns the uniqueness of Laurent series representations.

Theorem. If a series

^ (z - Zo)n

converges to f(z) at all points in some annular domain about z0, then it is the Laurent series expansion for f in powers of z — z0 for that domain.

The method of proof here is similar to the one used in proving Theorem. The hypothesis of this theorem tells us that there is an annular domain about z0 such that

f(z)=Cn(z — z0)n for each point z in it. Let g(z) be as defined by equation, but now allow n to be a negative integer too. Also, let C be any circle around the annulus, centered at Z0 and taken in the positive sense. Then, using the index of summation m and adapting to series involving both nonnegative and negative powers of z — Z0, write

f g(z)f(z)dz=V Cm (g(z)(z — Z0)m dz, y~~~r f /U dJ+l=Cm f 8{z){z - zof dz.

2*lJc(z — z0)n + 1 m=-«, JC

Since equations are also valid when the integers m and n are allowed to be negative, equation reduces to

flZUlZ ,.=0. ±1. ±2....).

2nl Jc (z — z0)n+1

which is expression for coefficients in the Laurent series for f in the annulus.

9.11 MULTIPLICATION AND DIVISION OF POWER SERIES

Suppose that each of the power series

^Zan(z - z0)^n and ^bn(z — z0)^n

converges within some circle $|Z - Z0| = R$. Their sums f(z) and g(z), respectively, product of those sums has a Taylor series expansion which is valid there:

f(z)g(z)=Y^cn(z — z0)n ($|Z - Z0|$< R).

C0=f(z0)g(z0)=a0 b0,

f(zo)g'izo) + f'(zo)g(zo) u ^c i =…..=a0bi + aibo,

and

f(zo)g''(zo) + 2f'(zo)g'(zo) + f''(zo)g(zo)cz = — = aobz + ci\b\ + azb.

The general expression for any coefficient cn is easily obtained by referring to Leibniz's rule

[f(z)g(z)]$^{(n)}$=C!)f(k)(z)g(n—k)(z) (n=1, 2,...),

where

/«\= _n. (jt=0,1,2 n),

for the nth derivative of the product of two differentiable functions. As usual, f(0)(z)=f(z) and 0!=1. Evidently,

and so expansion can be written

Notes

f(z)g(z)=a0b0 + (a0b1 + a1b0)(z - Z0)+ (a0b2 + a1b1 + aib0)(z - z0) +

+ (y] $^{a}k^{b}$n-k\ (z — z0)^{n} +'•• (|Z — Z0|< R)

Series is the same as the series obtained by formally multiplying the two series term by term and collecting the resulting terms in like powers of z — z0 ; it is called the Cauchy product of the two given series.

EXAMPLE . The function e^{z}/(1 + z) has a singular point at z=—1, and so its Maclaurin series representation is valid in the open disk |z| < 1. The first three nonzero terms are easily found by writing

e=e11- \ = (l+z + lrZ2 + l:Z3 -1 V1 ~ ^ + ::2 ~)

1 + z 1 - (-z) \ 2 6

and multiplying these two series term by term. To be precise, we may multiply each term in the first series, then each term in that series by -z, etc. The following systematic approach is suggested, where like powers of z are assembled vertically so that their coefficients can be readily added:

1 + z + -Z + + ...—z — z — -z — -z — …..

Z^3 + Z'^4 + -U5 + ---— z — z — -z — -z — •• •

The desired result is

ez/1+z

y—=1 + ~$_{Z2}$ -~Z3 + ... (|Z| < 1)

Continuing to let *f(z)* and *g(z)* denote the sums of series suppose that *g(z)* = 0 when |Z — Z0|< R. Since the quotient *f(z)/g(z)* is analytic throughout the disk |Z — Z0| < *R*, it has a Taylor series representation

= ^d_n(z-zo)" (\z-zo\<R)-

where the coefficients d_n can be found by differentiating *f(z)/g(z)* successively and evaluating the derivatives at *z* = z0. The results are the same as those found by formally carrying out the division of the first of series by the second. Since it is usually only the first few terms that are needed in practice, this method is not difficult.

EXAMPLE. As pointed out the zeros of the entire function sinhz are the numbers $z = nni$ $(n = 0, \pm1, \pm2,...)$. So the quotient

$$z^2 \sinh z \; z2(z + z^3/3! + z^5/5! + \cdots)$$

which can be written

$$z^2 \sinh z \; z^3 \backslash 1 + z2/3! + z^4/5! +$$

has a Laurent series representation in the punctured disk $0 < |z| < n$. The denominator of the fraction in parentheses on the right-hand side of equation is a power series that converges to (sinh $z)/z$ when $z = 0$ and to 1 when $z = 0$. Thus the sum of that series is not zero anywhere in the disk $|z| < n$;

Check your Progress-1

Discuss Series

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Discuss Integration And Differentiation Of Power Series

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9.12 LET US SUM UP

Notes

In this unit we have discussed the definition and example of Series, Convergence Of Sequence, Convergence Of Series, Taylor Series, Laurent Series, Absolute And Uniform Convergence Of Power Series, Continuity Of Sums Of Power Series, Integration And Differentiation Of Power Series, Uniqueness Of Series Representations, Multiplication And Division Of Power Series

9.13 KEYWORDS

Series This chapter is devoted mainly to series representations of analytic functions Convergence Of Sequence An infinite sequence z_1 ,z_2,---,Zn,...

Convergence Of Series An infinite series TO zn=z_1 + z_2 + ... + zn +...

Taylor Series The proof when z0 is arbitrary will follow as an immediate consequence

Laurent Series If a function f fails to be analytic at a point z0, one cannot apply Taylor's theorem at that point.

Absolute And Uniform Convergence Of Power Series This section and the three following it are devoted mainly to various properties of power series

Continuity Of Sums Of Power Series Our theorem is an important consequence of uniform convergence, discussed in the previous section.

Integration And Differentiation Of Power Series We have just seen that a power series O S(z) =y an(z — Z0)n

Uniqueness Of Series Representations The uniqueness of Taylor and Laurent series representations, anticipated respectively, follows readily from. We consider first the uniqueness of Taylor series representations.

Multiplication And Division Of Power Series Suppose that each of the power series ^Zan(z - z0)n and ^bn(z — z0)n

9.14 QUESTIONS FOR REVIEW

Explain Series

Explain Integration And Differentiation Of Power Series

9.15 ANSWERS TO CHECK YOUR PROGRESS

Series (answer for Check your Progress-1 Q)

Integration And Differentiation Of Power Series
(answer for Check your Progress-1 Q)

9.16 REFERENCES

- Complex Functions & Variables
- Complex Variables
- Introduction To Complex Analysis
- Application Of Complex Analysis & Variables
- The Complex Number System

UNIT-10 : RESIDUES AND POLES

STRUCTURE

10.0 OBJECTIVES

After studying this unit, you should be able to:

Learn, Understand about Residues And Poles

Isolated Singular Points

Residue

Cauchy's Residue Theorem

Residue At Infinity

Residues At Poles

Zeros Of Analytic Functions

Zeros And Poles

10.1 INTRODUCTION

In this part of the course we will study some basic complex analysis . This is an extremely useful and beautiful part of mathematics and forms the basis of many techniques employed in many branches of mathematic In this section we will study complex functions of a complex variable, Residues And Poles, Isolated Singular Points, Residue, Cauchy's Residue Theorem, Residue At Infinity, Residues At Poles, Zeros Of Analytic Functions, Zeros And Poles

10.2 RESIDUES AND POLES

The Cauchy-Goursat theorem states that if a function is analytic at all points interior to and on a simple closed contour C, then the value of the integral of the function around that contour is zero. If, however, the function fails to be analytic at a finite number of points interior to C, there is, as we shall see in this chapter, a specific number, called a residue, which each of those points contributes to the value of the integral. We develop here the theory of residues. we shall illustrate their use in certain areas of applied mathematics.

10.3 ISOLATED SINGULAR POINTS

Recall that a point Z0 is called a singular point of a function f if f fails to be analytic at Z0 but is analytic at some point in every neighborhood of Z0. A singular point Z0 is said to be isolated if, in addition, there is a deleted neighborhood $0 < |z — Z01 < e$ of Z0 throughout which f is analytic.

EXAMPLE. The function

Notes

z + 1 z3(z_2 + 1)

has the three isolated singular points z=0 and z=±i.

EXAMPLE . The origin is a singular point of the principal branch

*Log*z = *ln* r + iθ (r > *0,* — n < θ < n)

of the logarithmic function. It is not, however, an isolated singular point since every deleted e neighborhood of it contains points on the negative real axis and the branch is not even defined there. Similar remarks can be made regarding any branch

logz=ln r + id (r > 0, a < θ < a + 2n) of the logarithmic function.

EXAMPLE . The function

sin(n/z)

has the singular points z=0 and z=1/n (n=±1, ±2,...), all lying on the segment of the real axis from z=— 1 to z=1. Each singular point except z=0 is isolated. The singular point z=0 is not isolated because every deleted e neighborhood of the origin contains other singular points of the function. More precisely, when a positive number e is specified and m is any positive integer such that m > 1/e, the fact that 0 < 1/m < e means that the point z=1/m lies in the deleted e neighborhood 0 < |z| < e

In this chapter, it will be important to keep in mind that if a function is analytic everywhere inside a simple closed contour C except for a finite number of singular points

z_1, z_2, • • • ,

those points must all be isolated and the deleted neighborhoods about them can be made small enough to lie entirely inside C. To see that this is so, consider any one of the points zk. The radius e of the needed deleted neighborhood can be any positive number that is smaller than the distances to the other singular points and also smaller than the distance from zk to the closest point on C.

Finally, we mention that it is sometimes convenient to consider the point at infinity as an isolated singular point. To be specific, if there is a positive number R1 such that f is analytic for R1 < |z| < *x>, then f is said to have an isolated singular point at Z0 =*x>. Such a singular point will be use

10.4 RESIDUES

When Z0 is an isolated singular point of a function f, there is a positive number R2 such that f is analytic at each point z for which 0 < |z — Z0I < R2. Consequently, f (z) has a Laurent series representation

f (Z)=an(z ~ ^0)" H 1" " "2 +

z — z0 (z — z0)^2 (z — zo)^n

(0 < |z — zq| < R2),

where the coefficients an and bn have certain integral representations (Sec. 60). In particular,

* " f(z) dz 2ni Jc (z-z0)~$^{"+1}$

where C is any positively oriented simple closed contour around z0 that lies in the punctured disk 0 < |z — zo| < R2. When n=1, this xpression for bn becomes

f f(z)dz=2nib1.

The complex number b1, which is the coefficient of 1/(z — zo) in expansion is called the residue of f at the isolated singular point zo, and we shall often write

b1=Res f(z).

z=zo

Equation then becomes

f f(z) dz=2ni Res f(z).

as z=zo

Sometimes we simply use B to denote the residue when the function f and the point Z0 are clearly indicated.

Notes

Equation provides a powerful method for evaluating certain integrals around simple closed contours.

EXAMPLE . Consider the integral

J $_{Z1}$ sin^^ dz

where C is the positively oriented unit circle |z|=1. Since the integrand is analytic everywhere in the finite plane except at z=0, it has a Laurent series representation that is valid when 0 < |z| < rc>. Thus, according to equation the value of integral is 2ni times the residue of its integrand at z=0.

To determine that residue, we recall the Maclaurin series representation

z3 z5 z

smz =z- — + — - — + ... (|z| < 00)

and use it to write

:2s"lUJ=;-3!T + j!'?-7i'?+"

The coefficient of 1/z here is the desired residue. Consequently,

;2sin(dz=2ni

EXAMPLE. Let us show that j exp^zL^ dz=0

when C is the same oriented circle |z|=1 as in Example. Since 1/$_{Z2}$ is analytic everywhere except at the origin, the same is true of the integrand. The isolated singular point z=0 is interior to C, and can be used here as well. With the aid of the Maclaurin series representation

^=l + l! + 2! + 3!+'" (Iz| < 00), one can write the Laurent series expansion

exp(?)=1 + n'? + r? + r? + -' 10

The residue of the integrand at its isolated singular point z=0 is, therefore, zero (b1=0), and the value of integral is established.

We are reminded in this example that although the analyticity of a function within and on a simple closed contour C is a sufficient condition for the value of the integral around C to be zero, it is not a necessary condition.

EXAMPLE . A residue can also be used to evaluate the integral

c $z(z - 2)^4$

where C is the positively oriented circle |z — 2|=1. Since the integrand is analytic everywhere in the finite plane except at the points z=0 and z=2, it has a Laurent series representation that is valid in the punctured disk 0 < |z - 2|

2ni times the residue of its integrand at z=2. To determine that residue, we recall the Maclaurin series expansion

= £>" (LI < l)

and use it to write

$z(z - 2)^4$ $(z - 2)^4$ 2 + (z - 2) $2(z — 2)^4$ 1_(_z^2

^ (1)«

= EWr(;"2)""4 (0 < |z2| < 2).

In this Laurent series, which could be written in the form, the coefficient of 1/(z - 2) is the desired residue, namely -1/16. Consequently,

C $z(z - 2)^4$ V 16/ 8

10.5 CAUCHY'S RESIDUE THEOREM

If, except for a finite number of singular points, a function f is analytic inside a simple closed contour C, those singular points must be isolated The following theorem, which is known as Cauchy's residue theorem, is a precise statement of the fact that if f is also analytic on C and if C is positively oriented, then the Cauchy's Residue Theorem

Notes

value of the integral of f around C is 2ni times the sum of the residues of f at the singular points inside C. Let the points zk (k=1, 2,... ,n) be centers of positively oriented circles Ck which are interior to C and are so small that no two of them have points in common. The circles Ck, together with the simple closed contour C, form the boundary of a closed region throughout which f is analytic and whose interior is a multiply connected domain consisting of the points inside C and exterior to each Ck. Hence, according to the adaptation of the Cauchy-Goursat theorem to such domains

$$\int_C f(z)\,dz - \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = 0.$$

f f(z)dz = 2ni Res f(z) (k = 1, 2,...,n),

where C is the circle |z|=2, described counterclockwise. The integrand has the two isolated singularities z=0 and z=1, both of which are interior to C. We can find the residues B\ at z=0 and B2 at z=1 with the aid of the Maclaurin series

= 1 + z + Z2 + (|z| < 1).

We observe first that when 0 < |z| < 1 (Fig. 88),

5z — 2 5z — 2 —1 2\, ,, 2

= 5 (-1 - z - z);

z(z — 1) z 1 — z\z

and, by identifying the coefficient of 1/z in the product on the right here, we find that B1=2. Also, since

5z — 2 5(z — 1) + 3 1

z(z — 1) z — 1 1 + (z — 1)

= (5 + -^j[l-(z-l) + (z-l)^2

when 0 < |z — 1| < 1, it is clear that B2=3. Thus

5z — 2

dz=2ni(B1 + B2)=10ni.

C z(z — 1)

In this example, it is actually simpler to write the integrand as the sum of its partial fractions:

z(z - 1) z + z - 1'

Then, since 2/z is already a Laurent series when 0 < |z| < 1 and since 3/(z — 1) is a Laurent series when 0 < |z — 1| < 1, it follows that

C 5Z — 2

dz=2ni(2) + 2ni(2)=lOni.

10.6 RESIDUE AT INFINITY

Suppose that a function f is analytic throughout the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour c . Next, let R1 denote a positive number which is large enough that C lies inside the circle |z|=R1. The function f is evidently analytic throughout the domain R1 < |z| < and, as already mentioned at the end of the point at infinity is then said to be an isolated singular point of f.

Now let C0 denote a circle |z|=R0, oriented in the clockwise direction, where R0 > R1. The residue of f at infinity is defined by means of the equation

f (z) dz=2ni Res f(z).

z=to

Note that the circle C0 keeps the point at infinity on the left, just as the singular point in the finite plane is on the left in equation. Since f is analytic throughout the closed region bounded by C and C0, the principle of deformation

[f(z)dz=(f(z)dz=—[f (z) dz.

So, in view of definition

Notes

To find this residue, write the Laurent series

f(z)=CnZU (R < |Z| < rc),

where

c,=d-f («=0, ±1, ±2).

Replacing z by 1/z in expansion and then multiplying through the result by 1/Z2 we see that

n=—rc n=—rc x

and

Putting n=—1 in expression we now have

or

I f(z) dz=—2ni Res

Note how it follows from this and definition that

With equations, the following theorem is now established. This theorem is sometimes more efficient to use than Cauchy's residue theorem since it involves only one residue.

Theorem. If a function f is analytic everywhere in thefinite plane except for a finite number of singular points interior to a positively oriented simple closed contour C

EXAMPLE. Evaluated the integral of

f(z) =5z – 2 / z(z - 1)

around the circle |z| =2, described counterclockwise, by finding the residues of f(z) at z=0 and z=1. Since

-f(-) - 5 ~2z 5~2z 1

Z2 |z| z(1 - z) z 1 - z = - 2^)(1 + z + Z2 H)

= - + 3 + 3z + --- (0 < |z| < 1).

we see that the theorem here can also be used, where the desired residue is 5. More precisely, 5z - 2

dz=27TZ'(5)=107T*', C z(z - 1)

where C is the circle in question. This is, of course, the result obtained

EXAMPLE . Observe that the function

z^2 - 2z + 3 z(z - 2) + 3/(z-2)

= z H 3/(z-2)

2 + (z - 2) +3/(z-2)

(0 < \z - 2\ < to)

has a simple pole (m=1) at z_0=2. Its residue b_1 there is 3.

When representation is written in the form

f(z)=^ Cn(z - z0)^n (0 < \z - z0\< R2),

the residue of f at z0 is, of course, the coefficient c-1.

EXAMPLE . The function

sinh z 1/z3 z5 z7\111 z z3~=?^+3! + 5! + 7!+"7=? + 3!'Z + 5! + 7!+"'
(0 < \z\ < to)

has a pole of order m=3 at z0=0, with residue B=1/6.

There remain two extremes, the case in which every coefficient in the principal is zero and the one in which an infinie number of them are nonzero. When every bn is zero, so that

f(z)=^an(z - Z0)^n=a.0 + a\(z - Z0) + a2(z - Z0)^2 + (0 < $|Z - Z0|$ < R2),

z0 is known as a removable singular point. Note that the residue at a removable singular point is always zero. If we define, or possibly redefine, f at z0 so that f(z0)=a0, expansion becomes valid throughout the entire disk $|Z - Z0|$ < R2.

Notes

Since a power series always represents an analytic function interior to its circle of convergence, it follows that f is analytic at z0 when it is assigned the value a0 there. The singularity z0 is, therefore, removed.

EXAMPLE 5. We recall from (0 < |z| < m).

From this we see that e1/z has an essential singular point at z0=0, where the residue b1 is unity.

This example can be used to illustrate an important result known as Picard's theorem. It concerns the behavior of a function near an essential singular point and states that in each neighborhood of an essential singular point, a function assumes everyfinite value, with one possible exception, an infinite number of times.In the remaining sections of this chapter, we shall develop in greater depth the theory of the three types of isolated singular points just described. The emphasis will be on useful and efficient methods for identifying poles and finding the corresponding residues.

Exercise :

In each case, write the principal part of the function at its isolated singular point and determine whether that point is a pole, a removable singular point, or an essential singular point:

MzcJ^3-): W=-: (0^\z/ *1* + z z z *(2* - z)^3

Show that the singular point of each of the following functions is a pole. Determine the order m of that pole and the corresponding residue B.

Suppose that a function f is analytic at z0, and write g(z) = f(z)/(z — z0). Show that

(a) if f(z0) = 0, then z0 is a simple pole of g, with residue f(z0);

(b) if f(z0) = 0, then z0 is a removable singular point of g.

Suggestion: As pointed out in Sec. 57, there is a Taylor series for f(z) about z0 since f is analytic there. Start each part of this exercise by writing out a few terms of that series.

Use the fact (see Sec. 29) that e^z = — 1 when z = (2n + 1)ni (n = 0, ±1,

±2,...)

to show that $e^{1/z}$ assumes the value —1 an infinite number of times in each neighborhood of the origin. More precisely, show that $e^{1/z}$ = — 1 when

^{z}=~ $_{n}$ ' $_{n}$ (« = 0,±1,±2,...); (2n + 1)n

then note that if n is large enough, such points lie in any given e neighborhood of the origin. Zero is evidently the exceptional value in Picard's theorem

10.7 RESIDUES AT POLES

When a function f has an isolated singularity at a point z0 , the basic method for identifying z0 as a pole and finding the residue there is to write the appropriate Laurent series and to note the coefficient of 1/(z — z0). The following theorem provides an alternative characterization of poles and a way of finding residues at poles that is often more convenient.

Theorem. An isolated singular point z0 of a function f is a pole of order m if and only if f(z) can be written in the form

/•/\(P(z)

where $(z) is analytic and nonzero at z0 . Moreover,

Res f(z)=$(i0) if m=1 z=z0

and

Res f(z)=— &{m—1)(z0) / ^ ifm > 2. z=z0 (m — 1)!

Observe that expression need not have been written separately since, with the convention that 0(0)(z0)=$(i0) and 0!=1, expression reduces to it when m=1.

Notes

To prove the theorem, we first assume that f(z) has the form and recall that since $(z) is analytic at z0, it has a Taylor series representation

^ - », + + • - + - a,"-

in some neighborhood |z — z01 < £ of z0; and from expression it follows that

<P(zo) <p'(zt))/l\ <p"(zo)/2\ (zn)/(m - 1)!

J(Z)= TT^ + 77 _ ,,,,_i + TT _ + ... +

(z — z0)m (z — z0)m 1 (z — z0)m 2z — z0 + 2_ 1 ~ ^o)"

when 0 < |z — zo| < e. This Laurent series representation, together with the fact that $(zo)=0, reveals that zo is, indeed, a pole of order m of f(z). The coefficient of 1/(z — Z0) tells us, of course, that the residue of f(z) at Z0 is as in the statement of the theorem.

Suppose, on the other hand, that we know only that z0 is a pole of order m of f, or that f(z) has a Laurent series representation

f(z) =jran(z - zo)" + —— + ^ 2 + • • • + b'n

z — z0 (z — z0)2 (z — z0)m 1 (z — z0)m (bm=0)

which is valid in a punctured disk 0 < $|Z - Z0|$ < R2. The function $(z) defined by means of the equations

ch(z)=\(z — za)mf(z) when z=z0, \bm when z=z0

evidently has the power series representation

<P(z)=bm + bm —1(z — z0) + ... +b2(z — z0) +b1(z — z0) +^7an(z — z0)m

throughout the entire disk $|Z - Z0|$ < R2. Consequently, $(z) is analytic in that disk and, in particular, at z0. Inasmuch as $(z0)=bm=0, expression is established; and the proof of the theorem is complete.

EXAMPLE . The function

f(z) =$_{Z2}$ + 9

has an isolated singular point at z=3i and can be written

ft\^(z) u ,/, z + 1 J(z) = — where (p(z) = z — 3i z + 3i

Since $(z) is analytic at z=3i and $(3i)=0, that point is a simple pole of the function f; and the residue there is

Bl=<P(3i)=— 3i + 1 —i 3 — i r=z • 6

The point z=—3i is also a simple pole of f, with residue

Bz=——.3+i/6

EXAMPLE. If

-3 + 2-

f(Z)=I T3' (z — i)3

then

f(z) = where 4>(z)=ti3 + 2z./ (z — i)3

The function $(z) is entire, and $(i)=i=0. Hence f has a pole of order 3 at z=i, with residue

4>" (i) 6i

B=~2T=2!=u

The theorem can, of course, be used when branches of multiple-valued functions are involved.

EXAMPLE. Suppose that

f(Z)=2 _L 1 '

z_2 + 1

where the branch

logz=lnr + id (r > 0, 0 < d < 2n)

of the logarithmic function is to be used. To find the residue of f at the singularity z=i, we write

$(z) $(\log z)^3$

Notes

f(z) = r where <p(z) = —.z — i z + i

The function $(z) is clearly analytic at z=i; and, since

$(\log i)^3$ $(\ln 1 + i\pi/2)^3$ n^3

<p(i)=8= 1—— = + 0,

f has a simple pole there. The residue is

B=<p(i)=- —.$n^3/16$

While the theorem can be extremely useful, the identification of an isolated singular point as a pole of a certain order is sometimes done most efficiently by appealing directly to a Laurent series.

EXAMPLE. If, for instance, the residue of the function

$f(z) = \sinh z / z^4$

is needed at the singularity z=0, it would be incorrect to write

f(z)=—j- where <p(z)=sinfu

and to attempt an application of formula with m=4. For it is necessary that 0(z0)=0 if that formula is to be used. In this case, the simplest way to find the residue is to write out a few terms of the Laurent series for f(z), There it was shown that z=0 is a pole of the third order, with residue B=1/6.

In some cases, the series approach can be effectively combined with the theorem

10.8 ZEROS OF ANALYTIC FUNCTIONS

Zeros and poles of functions are closely related. In fact, we shall see in the next section how zeros can be a source of poles. We need, however, some preliminary results regarding zeros of analytic functions.

Suppose that a function f is analytic at a point zo. We know that all of the derivatives f(n)(z) (n=1, 2,...) exist at zo. If f(zo)=0 and if there is a positive integer m such that f(m)(zo)=0 and each derivative of lower order vanishes at zo, then f is said to have a zero of order m at zo. Our first theorem here provides a useful alternative characterization of zeros of order m.

Theorem. Let a function f be analytic at a point zo • It has a zero of order m at zo if and only if there is a function g, which is analytic and nonzero at zo , such that

f(z)=(z — zo)mg(z).

Both parts of the proof that follows use the fact that if a function is analytic at a point zo, then it must have a Taylor series representation in powers of z — zo which is valid throughout a neighborhood |z — zol < s of zo.

We start the first part of the proof by assuming that expression holds and noting that since g(z) is analytic at zo , it has a Taylor series representation

, > , > , g'(zo), , , g"(Zo), ,2 , g(z) — y(zo)H j-j—(z — zoH ^—(z — zo) +...

in some neighborhood $|Z - Z0|$ < s of zo. Expression thus takes the form

/(z)=g(zo)(z - zo)'" + - Z0)'n+1 + - Z0)'"+2 + • • •

when |z — zo| < s. Since this is actually a Taylor series expansion for f(z), according to Theorem it follows that and that

= m\g(zo)=0.

Hence zo is a zero of order m of f.

Conversely, if we assume that f has a zero of order m at zo, the analyticity of f at Z0 and the fact that conditions hold tell us that in some neighborhood

$|Z - Z0|$ < e, there is a Taylor series

fl , ^f"Hz0) f(z)=2^ —i—^ - ^o)

f(,n)(z0) fl»'+h{zo) /('"+2)(Z0), >2 ,

i ^ ~(—1 111 (z — zo) H—-— „ . (z - so) + m\ (m + 1)\ (m + 2)\

Consequently, f(z) has the form, where

, , fim)(zo) , fim+h(zo), , , fim+2)(Z0), >2 ,

g(z) — : 1 . (z — zo) + — . (z — zo) +

m\ (m + 1)\ (m + 2)\ (|z — zo I < e).

The convergence of this last series when |z — zo I < e ensures that g is analytic in that neighborhood and, in particular, at zo Moreover,

, , fim)(z 0) g(zo) = j— ± 0.

This completes the proof of the theorem.

EXAMPLE . The polynomial $f(z)=z^3 — 8=(z — 2)(z^2 + 2z + 4)$ has a zero of order m=1 at zo=2 since

f(z)=(z — 2)g(z),

where $g(z)=z^2 + 2z + 4$, and because f and g are entire and g(2)=12=o. Note how the fact that zo=2 is a zero of order m=1 of f also follows from the observations that f is entire and that

f (2)=o and f '(2)=12=o.

EXAMPLE . The entire function f(z)=z(ez — 1) has a zero of order m=2 at the point zo=o since

f (o)=f '(o)=o and f'' (o)=2=o.

In this case, expression becomes

f(z)=(z — oyLg(z),

where g is the entire function defined by means of the equations

g(7)=l(eZ - 1)/z when z=0,

g(7) 11 when z=0.

Our next theorem tells us that the zeros of an analytic function are isolated when the function is not identically equal to zero.

Theorem. Given a function f and a point 70 , suppose that

f is analytic at 70 ;

f(z0)=0 but f(z) is not identically equal to zero in any neighborhood of 70 . Then f(z)=0 throughout some deleted neighborhood $0 < |Z — 70| < s$ of 70 •

To prove this, let f be as stated and observe that not all of the derivatives of f at 70 are zero. If they were, all of the coefficients in the Taylor series for f about 70 would be zero; and that would mean that f(y) is identically equal to zero in some neighborhood of 70 . So it is clear from the definition of zeros of order m at the beginning of this section that f must have a zero of some finite order m at 70. According to Theorem 1, then,

f(7)=(7 — 70)mg(7)

where g(7) is analytic and nonzero at 70 .

Now g is continuous, in addition to being nonzero, at 70 because it is ana- lytic there. Hence there is some neighborhood $|Z — 70| < s$ in which equation holds and in which g(7)=0 (see Sec. 18). Consequently, f(7)=0 in the deleted neighborhood $0 < |Z — 70| < s$; and the proof is complete.

Our final theorem here concerns functions with zeros that are not all isolated. It was referred to earlier in Sec. 27 and makes an interesting contrast to Theorem.

Theorem. Given a function f and a point 70 , suppose that

f is analytic throughout a neighborhood N0 of 70 ;

f(7)=0 at each point 7 of a domain D or line segment L containing 70
Then f(7)=0 in N0; that is, f(7) is identically equal to zero throughout

We begin the proof with the observation that under the stated conditions, f (7)=0 in some neighborhood N of 70. For, otherwise, there would be a

Notes

deleted neighborhood of 70 throughout which f(7)=0, according to Theorem; and that would be inconsistent with the condition that f(7)=0 everywhere in a domain D

or on a line segment L containing zo . Since f(z)=0 in the neighborhood N, then, it follows that all of the coefficients

an= j (/?=0, 1, 2)

in the Taylor series for f (z) about Z0 must be zero. Thus f(z)=0 in the neighborhood N0, since the Taylor series also represents f(z) in N0. This completes the proof.

10.9 ZEROS AND POLES

The following theorem shows how zeros of order m can create poles of order m.

Theorem . Suppose that

two functions p and q are analytic at a point z0 ;

p(z0)=0 and q has a zero of order m at z0 •

Then the quotient p(z)/q(z) has a pole of order m at z0 •

The proof is easy. Let p and q be as in the statement of the theorem. Since q has a zero of order m at z0, we know from Theorem that there is a deleted neighborhood of z0 throughout which q(z)=0; and so z0 is an isolated singular point of the quotient p(z)/q(z)• Theorem tells us, moreover, that

$q(z)=(z - z0)^{m}g(z)$, where g is analytic and nonzero at z0 ; and this enables us to write

£!£>= Where ^,=£!±2. q(z) $(Z-Zo)^{m}$ g(Z)

Since $(z) is analytic and nonzero at z0 , it now follows from the theorem in that z0 is a pole of order m of p(z)/q(z).

EXAMPLE . The two functions

p(z)=1 and q(z)=z(ez - 1)

are entire; and we know from Example that q has a zero of order m=2 at the point Z0=0. Hence it follows from Theorem that the quotient

plz)=1 q(z) Z(ez - 1)

has a pole of order 2 at that point. This was demonstrated in another way in Theorem leads us to another method for identifying simple poles and finding the corresponding residues. This method, stated just below as Theorem, is sometimes easier to use than the theorem

Theorem. Let two functions p and q be analytic at a point Z0 If P(Z0)=0, q(Z0)=0, and q'Z)=0, then Z0 is a simple pole of the quotient p(z)/q(z) and

Res = p(z) p(Z0) / Z=Z0 q(z) q'(Z0)

To show this, we assume that p and q are as stated and observe that because of the conditions on q, the point z0 is a zero of order m=1 of that function. According then,

q(z)=(z - Z0)g(z)

where g(z) is analytic and nonzero at Z0- Furthermore, Theorem in this section tells us that Z0 is a simple pole of p(z)/q(z); and expression for p(z)/q(z) in the proof of that theorem becomes

p(z)/q(z) = $(z) / z - Z0 where <p(z) =p(z)/f(z)

Since this $(z) is analytic and nonzero at Z0, we know from the theorem that

Resrl7=rlM. z=z0 q(z) g(Z0)

But g(z0)=q'(Z0), as is seen by differentiating each side of equation and then setting z=Z0- Expression thus takes the form.

EXAMPLE. Consider the function

f(z)=cot 7 = cos z / sin z

which is a quotient of the entire functions p(z)=cosz and q(z)=sinz. Its singularities occur at the zeros of q, or at the points

z=nn (n=0, ±1, ±2,...).

Notes

Since

p(nn)=(-1)^n=0, q(nn)=0, and q '(nn)=(— 1)^n=0, each singular point z=nn of f is a simple pole, with residue

D p(ri7T) (-1)" 1 tSyt - X.

q'(nn) (— 1)"

Although this residue can also be found by the method in the computation is somewhat more involved. There are formulas similar to formula for residues at poles of higher order, but they are lengthier and, in general, not practical.

EXERCISES

- Show that the point z=0 is a simple pole of the function

f(z)=cscz = 1/sin z

and that the residue there is unity by appealing to

Laurent series for cscz that was found in Exercise

- Show that

z - sinh z i (a) Res , =

z=m $_{Z2}$ sinh z n

lb) Rest*>+ Res 55M=-2„S(«>.

z=ni sinh z z=-ni sinh z

- Show that

Res(^sec^)=(—1)"+1 where=—+«jr (/;=0, ±1, ±2, ...);

z=zn 2

Res(tanlu)=1 where z„=(— i (n=0, ±1, ±2,...).

z=zn V 2 '

- Let C denote the positively oriented circle |z|=2 and evaluate the integral

/t\1 dz

tanz dz:

(a) f tan^ dz: (b) f —- Jc Jc sir

'C JC sinh 2z

Ans. (a) —4ni; (b) —ni.

- Let CN denote the positively oriented boundary of the square whose edges lie along

the lines / x / x

,r=±(iV + -jjr and y=±(iV+-jjr,where N is a positive integer. Show that

dz

= 2ni

CN $_{Z2}$ sin z Then, using the fact that the value of this integral tends to zero as N tends to infinity point out how it follows that

T. (-1)',+1 jr2

~ IT

n=1

- Show that

dz n

I

c ($_{Z2}$ — l)2 + 3 2</2 '

where C is the positively oriented boundary of the rectangle whose sides lie along the lines x=±2, y=0, and y=1.

Suggestion: By observing that the four zeros of the polynomial q{z)={$_{Z2}$ — l)2 + 3 are the square roots of the numbers 1 ± </3/, show that the reciprocal 1 /q (z) is analytic inside and on C except at the points

</3 + / — </3 + ;

Notes

zo=^r an" =

Then apply Theorem.

- Consider the function

f(z) = 1 / [q(z)}2

where q is analytic at z0, q(z0)=0, and q'(z0)=0. Show that Z0 is a pole of order m=2 of the function f, with residue

q" (z0)

[q '(z0)]3

Suggestion: Note that z0 is a zero of order m=1 of the function q, so that q(z)=(z - z0)g(z) where g(z) is analytic and nonzero at z0. Then write

/U)=, where </>(z)=1

(z - z0)2 [g(z)]2

The desired form of the residue B0=$'(z0) can be obtained by showing that

q '(z0)=g(z0) and q "(z0)=2g'(z0).

- Use the result in Exercise to find the residue at z=0 of the function

(a) f(z)=csc2 z; (b) f(z) =

(z + Z2)2 Ans. (a) 0; (b) -2.

- Let p and q denote functions that are analytic at a point z0 , where p(z0)=0 and q(z0)=0. Show that if the quotient p(z)/q(z) has a pole of order m at z0 , then z0 is a zero of order m of q.

p(z) _ <p(z) q(z) (Z-Zo)"r where 0(z) is analytic and nonzero at zo . Then solve for q(z).

- Recall that a point Z0 is an accumulation point of a set S if each deleted neighborhood of Z0 contains at least one point of S. One form of the Bolzano- Weierstrass theorem can be stated as follows: an infinite set of points lying in a closed bounded region

R has at least one accumulation point in R.4 Use that theorem and Theorem to show that if a function f is analytic in the region R consisting of all points inside and on a simple closed contour C, except possibly for poles inside C, and if all the zeros of f in R are interior to C and are of finite order, then those zeros must be finite in number.

- Let R denote the region consisting of all points inside and on a simple closed contour C. Use the Bolzano-Weierstrass theorem and the fact that poles are isolated singular points to show that if f is analytic in the region R except for poles interior to C, then those poles must be finite in number.

BEHAVIOR OF FUNCTIONS NEAR ISOLATED SINGULAR POINTS

As already indicated the behavior of a function f near an isolated singular point zo varies, depending on whether zo is a pole, a removable singular point, or an essential singular point. In this section, we develop the differences in behavior somewhat further. Since the results presented here will not be used elsewhere in the book, the reader who wishes to reach applications of residue theory more quickly may pass directly to without disruption.

Theorem . If zo is a pole of a function f, then

lim f(z)=to.

To verify limit, we assume that f has a pole of order m at zo and use the theorem. It tells us that

f(z) = ft\^{z) / (z - zo)m

where $(z) is analytic and nonzero at zo. Since

i (z .n)m lim(z - zo)m

T 1 T $^{(z - zo)}$ z^zo n

.

Notes

lirri = lirri = = = 0,

z^zo f(z) z^zo (p(z) lim (p(z) (p(zo)

then, limit holds, according to the theorem in regarding limits that involve the point at infinity.

The next theorem emphasizes how the behavior of f near a removable singular point is fundamentally different from behavior near a pole.

Theorem Ifz0 is a removable singular point of a function f, then f isanalytic and bounded in some deleted neighborhood $0 < |Z — Z0| <$ £ of Z0 .

The proof is easy and is based on the fact that the function f here is analytic in a disk $|Z — Z0|$ < R2 when f(zo) is properly defined; f is then continuous in any closed disk$|Z — Z0|$ <£ where £ < R2. Consequently, f is bounded in that disk, and this means that, in addition to being analytic, f must be bounded in the deleted neighborhood 0 < $|Z — Z0|$< £.

The proof of our final theorem, regarding the behavior of a function near an essential singular point, relies on the following lemma, which is closely related to Theorem and is known as Riemann's theorem.

Lemma. Suppose that a function f is analytic and bounded in some deleted neighborhood 0 < $|Z — Z0|$ < £ of a point z0 . If f is not analytic at z0 , then it has a removable singularity there.

To prove this, we assume that f is not analytic at zo. As a consequence, the point z0 must be an isolated singularity of f; and f(z) is represented by a Laurent series throughout the deleted neighborhood 0 < $|Z — Z0|$ <£. If C denotes a positively oriented circle |z — zq|=p, where p < £ that the coefficients bn in expansion can be written

bn=— [f(Z) dZ (n=1,2,...). 2niJc (z-zo)~"+1

Now the boundedness condition on f tells us that there is a positive constant M such that |f (z)| < M whenever $0 < |Z - Z0| < e$. Hence it follows from expression

that

\b,,\ < 2np=Mp" (n=1,2,...). 2n p n+1

Since the coefficients bn are constants and since p can be chosen arbitrarily small, we may conclude that bn=0 (n=1, 2,...) in the Laurent series. This tells us that Z0 is a removable singularity of f, and the proof of the lemma is complete.

We know from that the behavior of a function near an essential singular point is quite irregular. The next theorem, regarding such behavior, is related to Picard's theorem in that earlier section and is usually referred to as the Casorati Weierstrass theorem. It states that in each deleted neighborhood of an essential singular point, a function assumes values arbitrarily close to any given number.

Theorem. Suppose that Z0 is an essential singularity of a function f, and let W0 be any complex number. Then, for any positive number e, the inequality

\f(z) — W0| < e

is satisfied at some point z in each deleted neighborhood $0 < |Z - Z0|$

<8 of Z0

assume that condition (4) is not satisfied for any point z there. Thus \f(z) - W0i> £ when 0 < \z — Z0\<5; and so the function

g(z)=——^ $\quad (0 < |Z - Z0| < 5)$

is bounded and analytic in its domain of definition. Hence, according to our lemma, Z0 is a removable singularity of g; and we let g be defined at Z0 so that it is analytic there.

If g(z0)=0, the function f(z), which can be written

f(z)=1/g(z0) + w0

Notes

when $0 < |z - z_0| < \delta$, becomes analytic at z_0 when it is defined there as

$$f(z_0) = \frac{1}{g(z_0)} + w_0.$$

But this means that z_0 is a removable singularity of f, not an essential one, and we have a contradiction.

If $g(z_0)=0$, the function g must have a zero of some finite order m at z_0 because g(z) is not identically equal to zero in the neighborhood $|z - z_0| < \delta$. In view of equation, then, f has a pole of order m at z_0

Check your Progress-1

Discuss Residues And Poles

Discuss Zeros And Poles

10.10 LET US SUM UP

In this unit we have discussed the definition and example of Residues And Poles, Isolated Singular Points, Residue, Cauchy's Residue Theorem, Residue At Infinity, Residues At Poles, Zeros Of Analytic Functions, Zeros And Poles

10.11 KEYWORDS

Residues And Poles.. The Cauchy-Goursat theorem states that if a function is analytic at all points interior to and on a simple closed contour C

Isolated Singular Points Recall that a point Z0 is called a singular point of a function f if f fails to be analytic at Z0 but is analytic at some point in every neighborhood of Z0

Residue When Z0 is an isolated singular point of a function f, there is a positive number R2 such that f is analytic at each point z for which 0 < |z — Z0I < R2.

Cauchy's Residue Theorem If, except for a finite number of singular points, a function f is analytic inside a simple closed contour C,

Residue At Infinity Suppose that a function f is analytic throughout the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour c .

Residues At Poles When a function f has an isolated singularity at a point z0 , the basic method for identifying z0 as a pole and finding the residue there is to write the appropriate Laurent series

Zeros Of Analytic Functions Zeros and poles of functions are closely related. In fact, we shall see in the next section how zeros can be a source of poles. We need, however, some preliminary results regarding zeros of analytic functions

Zeros And Poles The theorem shows how zeros of order m can create poles of order m.

10.12 QUESTIONS FOR REVIEW

Explain Residues And Poles

Explain Zeros And Poles

10.13 ANSWERS TO CHECK YOUR PROGRESS

Residues And Poles (answer for Check your Progress-1 Q)

Zeros And Poles (answer for Check your Progress-1 Q)

10.14 REFERENCES

- Complex Functions & Variables
- Complex Variables
- Complex Functions
- Complex Numbers & Analysis
- The Complex Number System

UNIT-11 : APPLICATIONS OF RESIDUES

STRUCTURE

11.0 OBJECTIVES

After studying this unit, you should be able to:

Learn, Understand about Applications Of Residues

Evaluation Of Improper Integrals

Improper Integrals From Fourier Analysis

Jordan's Lemma

Integration Along A Branch Cut

Definite Integrals Involving Sines And Cosines

Argument Principle

Rouche's Theorem

11.1 INTRODUCTION

In this part of the course we will study some basic complex analysis . This is an extremely useful and beautiful part of mathematics and forms the basis of many techniques employed in many branches of mathematic In this section we will study complex functions of a complex variable, Applications Of Residues, Evaluation Of Improper Integrals, Improper Integrals From Fourier Analysis, Jordan's Lemma, Integration Along A Branch Cut, Definite Integrals Involving Sines And Cosines, Argument Principle, Rouche's Theorem

11.2 APPLICATIONS OF RESIDUES

We turn now to some important applications of the theory of residues, which was developed. The applications include evaluation of certain types of definite and improper integrals occurring in real analysis and applied mathematics. Considerable attention is also given to a method, based on residues, for locating zeros of functions and to finding inverse Laplace transforms by summing residues.

11.3 EVALUATION OF IMPROPER INTEGRALS

In calculus, the improper integral of a continuous function f(x) over the semi- infinite interval $0 < x <$ to is defined by means of the equation

When the limit on the right exists, the improper integral is said to converge to that limit. If f(x) is continuous for all x, its improper integral over the infinite interval —to < x < to is defined by writing and when both of the limits here exist, we say that integral converges to their sum. Another value that is assigned to integral is often useful. Namely, the Cauchy principal value (P.V.) of integral is the number provided this single limit exists.

If integral converges, its Cauchy principal value exists; and that value is the number to which integral converges these last two limits are the same as the limits on the right in equation

But suppose that f(x) (—to < x < to) is an even function, one where

f(—x)=f(x) for allx,and assume that the Cauchy principal value exists. The symmetry of the graph of y=f(x) with respect to the y axis tells us

f(x)=p(x)/q(x), where p(x) and q(x) are polynomials with real coefficients and no factors in common. We agree that q(z) has no real zeros but has at least one zero above the real axis.

The method begins with the identification of all the distinct zeros of the polynomial q(z) that lie above the real axis. They are, of course, finite in number and may be labeled z_1, z_2, ..., zn, where n is less than or equal to the degree of q(z). We then integrate the quotient

p(z)

f(z) =q(z)

around the positively oriented boundary of the semicircular region

That simple closed contour consists of the segment of the real axis from z=—R to z=R and the top half of the circle |z|=R, described counterclockwise and denoted by CR. It is understood that the positive number R is large enough so that the points z\,zz,..., zn all lie inside the closed path.

The parametric representation z=x(—R < x < R) of the segment of the real axis just mentioned and Cauchy's residue theorem can be used to write

Notes

f (x) dx + f f(z)dz=2ni Res f(z), R Jcr rr z=zk

f (x) dx=2ni Res f(z) — I f(z)dz. R rT z=« Jcr

lim f f(z)dz=0,

R^^J CR

it then follows that

CO n

f(x) dx=2ni Y^ Resf (z);

and if f(x) is even, equations

f(x) dx=2ni Y^ Resf(z)

o k=1 z=zk

and

f f(x)dx=ni Y^ Res f(z).

Example : we start with the observation that the function

f(z) = $_{Z2}$ / z6 + 1

has isolated singularities at the zeros of z6 + 1, which are the sixth roots of -1, and is analytic everywhere else. The method for finding roots of complex numbers reveals that the sixth roots of — 1 are

and it is clear that none of them lies on the real axis. The first three roots,

c0=e$^{in/6}$, c1=i, and c2=e$^{i5n/6}$,

lie in the upper half plane and the other three lie in the lower one. When R > 1, the points ck (k=0, 1, 2) lie in the interior of the semicircular region bounded by the segment z=x (-R < x < R) of the real axis and the upper half CR of the circle |z|=R from z=R to z=—R. Integrating f(z) counterclockwise around the boundary of this semicircular region, we see that

f f(x)dx + f f(z)dz=2ni(B0 + B1 + B2), where Bk is the residue of f(z) at ck (k=0, 1, 2).

We find that the points ck are simple poles of f and that

Bk=Res -f—=-L =   (*=0,1,2).

z=ck z^6 + 1 6c5 6c3

Thus

2ni(Bo + B\ + Bi)=2ni — — + —^=—; and equation can be put in the form

f f(x)dx=^~ — f f(z)dz,

which is valid for all values of R greater than 1.

Next, we show that the value of the integral on the right in equation tends to 0 as R tends to to. To do this, we observe that when |z|=R,

$|z2|=|z|^2=R^2$

and

$|z^6 + 1| >||z|^6 - 1|=R^6 - 1.$

So, if z is any point on CR,

|z2| R^2 1/(^)1=7"7   77 - Mr where Mr =|z6 + 1| - R R^6 - 1:

and this means that

f(z) dz

nR being the length of the semicircle CR. Since the number

nR3 MrtzR =R6 1

is a quotient of polynomials in R and since the degree of the numerator is less than the degree of the denominator, that quotient must tend to zero as R tends to to. More precisely, if we divide both numerator and enominator by R6 and write

Mr nR = 1_F

it is evident that MR nR tends to zero. Consequently, in view of inequality

Notes

lim/f(z) dz=0.

R^TO-/ Cr

It now follows from equation that

- Use residues to find the Cauchy principal values of the integrals in Exercises
- Use a residue and the contour where R > 1, to establish the integration formula

(k=0, I, 2,...,n - Iand that there are none on that axis.

- show thatz2m I

Res -i- = e"(2k+1)a (k=0, 1, 2 n - 1)

z=ck z + I 2n

where ck are the zeros found in part (a) and

2m + I

a=— jr.

2n

Then use the summation formula

n — I I _ zn

E^A"=—7 k=0 - z

to obtain the expression

n—I

2ni\^ Res

' J 7=C1,

z=ck z2n + I n sin a

k=0

- Use the final result in part (b) to complete the derivation of the integration formula.

[(2a2 3)VA. a a*JA. — a\,

[(x2 - a)2 + 1] 2 8V2A3

where a is any real number and A=s/ a2 + 1, arises in the theory of case-hardening of steel by means of radio-frequency heating.5 Follow the steps below to derive it.

- Point out why the four zeros of the polynomial

q(z)=(z2 - a)2 + 1 are the square roots of the numbers a ± i. Then, using the fact that the numbers

Zo=—7= (~F a ~F ' s/ A — a)

V 2

and — zo are the square roots of a + i verify that ±zo are the square roots of a — i and hence that zo and —zo are the only zeros of q(z) in the upper half plane Im z > 0.

- Using the method derived and keeping in mind that z(j=a + i for purposes of simplification, show that the point Z0 in part (a) is a pole of order 2 of the function f(z)=1/[q(z)]2 and that the residue B1 at Z0 can be written

_ ci"Co) _ a - i (2a2 + 3)

1 k/(zo)]3 16A2zo '

After observing that q'{—z)=— q'(z) and q"(—z)=q"(z), use the same method to show that the point —zo in part (a) is also a pole of order 2 of the function f(z), with residue

q"{zo)) = -i>i-[q'(z0)]3

Then obtain the expression—a + i(2a2 + 3)

Notes

Im z0for the sum of these residues.

(c) Refer to part (a) and show that |q(z)| > (R — |z0|)4 if |z|=R, where R > |z0|. Then, with the aid of the final result in part (b), complete the derivation of the integration formula.

11.4 IMPROPER INTEGRALS FROM FOURIER ANALYSIS

Residue theory can be useful in evaluating convergent improper integrals of the form

f(x) sin ax dx or/f(x) cos axdx,

where a denotes a positive constant. we assume that f(x)=p(x)/q(x) where p(x) and q(x) are polynomials with real coefficients and no factors in common. Also, q(x) has no zeros on the real axis and at least one zero above it. Integrals of type occur in the theory and application of the Fourier integral. The method described cannot be applied directly here since

|sin az\2=sin2 ax + sinh2 ay

and

|cos az|2=cos2 ax + sinh2 ay.

More precisely, since

eay _ e-ay

sinh av = - ,

the moduli | sin az| and |cos az| increase like eay as y tends to infinity. The modification illustrated in the example below is suggested by the fact that

/

R r R rR

f(x) cos ax dx + i I f(x) sin ax dx=I f(x)eiax dx, R -R -R

together with the fact that the modulus

|eiaz|=|eia(x+iy)|=|e-ayeiax|=e-ay

is bounded in the upper half plane y > 0.

EXAMPLE. Let us show

cos 3x 2n

$(.v_2 + l)^2$

Because the integrand is even, it is sufficient to show that the Cauchy principal value of the integral exists and to find that value.

We introduce the function

$f(z) = (_{Z2} + 1)^2$

and observe that the product f(z)ei3z is analytic everywhere on and above the real axis except at the point z=i. The singularity z=i lies in the interior of the semicircular region whose boundary consists of the segment — R < x < R of the real

axis and the upper half CR of the circle |z|=R (R > 1) from z=R to z=—R Integration of $f(z)e^{l3z}$ around that boundary yields the equation

R e^{l3X} p

dx=2,tZ1B\ - $f(z)e^{l3z}$ dz . R (x + 1)

where

B1=Res $[f(z)e^{l3z}]$

Since

&(z) where $(z) = (z — i)2 (z + i)2'

the point z=l is evidently a pole of order m=2 of f(z)el 3z; and

Bi=<p'(i)=^3. ie3

By equating the real parts on each side of equation then, we find that

Notes

R cos 3x z , 1 zzdx=~T ~re f f^e'3z dz-—R (x2 + 1)2 e3 .. c2 + 1)2

Finally, we observe that when z is a point on CR

|/U)| < Mr where Mr = $(R_2 — 1)^2$

and that |el3z|=e—3y < 1 for such a point. Consequently

11.5 JORDAN'S LEMMA

In the evaluation of integrals of the type treated, it is sometimes necessary to use Jordan's lemma * which is stated just below as a theorem.

Theorem. Suppose that a function f(z) is analytic at all points in the upper half plane y > 0 that are exterior to a circle |z|=R0;

Cr denotes a semicircle z=Re'e(0 < 6 < n), where R > R0 ;

for all points z on Cr, there is a positive constant Mr such that

lf(z)l < Mr and lim MR=0.

R^ro

Then, for every positive constant a,

lim f f (z)eiaz dz=0.

R^roJ CR

The proof is based on Jordan's inequality:

P~Rsine df) c

To verify it, we first note from the graphs of the functions

y=sin 9 and v=—

that

sin 6 > — when 0 < 9 < —.

Consequently, if R > 0,

e-Rsm8 < -2R6/x when Q < g < *

and so rn/2 rn/2 n

e~Rsm-d9< e~2R-/7T d9=— (1 -e~R) (R > 0).

Hence

I e-Rsined9<— (R> 0).

But this is just another form of inequality, since the graph of y=sin 6 is symmetric with respect to the vertical line 6=n/2 on the interval 0 < 6 < n. Turning now to the proof of the theorem, we accept statements (a)- (c) there and write

[f(z)eiazdz =[f(Rei6) exp(iaRei6)Riei6 d6. and in view of Jordan's inequality, it follows that

f (z)e dz>cr

The final limit in the theorem is now evident since MR ^ 0as R ^tx>.

EXAMPLE. Let us find the Cauchy principal value of the integral

x sin xdx $c^2 + 2x + 2$

As usual, the existence of the value in question will be established by our actually finding it.

We write

f,=z=z ~ z_2 + 2z + 2 (z — z_1) (z -z_l)'

where z_1 =— 1 + i. The point z_1, which lies above the x axis, is a simple pole of the function f(z)eiz, with residue

Bl=^=7i e^{iz1} / zt - z_1

Hence, when R > \fl and CR denotes the upper half of the positively oriented circle |z|=R, and that |eiz|=e—y < 1 for such a point. By proceeding as we did in the examples, we cannot conclude that the right-hand side of inequality, and hence its left-hand side, tends to zero as R tends to infinity. For the quantity does not tend to zero. The above theorem does, however, provide the desired limit, namely

Notes

lim/f(z)elzdz=0,

since

Mr = — ^ —> 0 as R 00 .

So it does, indeed, follow from inequality that the left-hand side there tends to zero as R tends to infinity. Consequently, equation, together with expression for the residue B1, tells us that

x sin x uxn =Im(27r/fii)=— (sin 1 + cos 1).

EXAMPLE. Modifying the method used we derive here the integration formula

,, s sinx n

I dx=-J0

by integrating eiz/z around the simple closed contour.p and R denote positive real numbers, where p < R; and L1 and L2 represent the intervals

P < x < R and — R < x < —p,

respectively, on the real axis. While the semicircle CR, the semicircle Cp is introduced here in order to avoid passing through the singularity z=0 of the quotient elz/z.

The Cauchy-Goursat theorem tells us that

[— dz+f — dz+f — dz+f Jl\ z Jcr z Jl2 z Jc

or

f eiz f eiz f eiz f eiz

I — dz+ — dz=— I — dz — I — dz. JL\ z JL2 z P Cp z JCr z

Moreover, since the legs L1 and — L2 have parametric representations

z=rel°=r (p < r < R) and z=rel7t=—r(p < r < R),respectively, the left-hand side of equation can be written

c piz f piz fR Pir fR p-ir f

I — dz — I — dz=I — dr — I -— dr=2i I

Jli z J-l2 z Jp r Jp r Jp L z

Consequently,

R sin r piz piz

2i I dr=— I — dz — I

Jp r Jcn z Jc

Now, from the Laurent seriesr representation

It is clear that pizlz has a simple pole at the origin, with residue unity. So, according to the theorem at the beginning of this section,

lim I — dz=—ni.

p^0JCp z

Also, since when z is a point on CR, we know from Jordan's lemma that

lim/— dz=0.

R^™JcR z Thus, by letting p tend to 0 in equation and then letting R tend to to, we arrive at the result

„ sin r 2i I dr=ni.

11.6 INTEGRATION ALONG A BRANCH CUT

Cauchy's residue theorem can be useful in evaluating a real integral when part of the path of integration of the function f(z) to which the theorem is applied lies along a branch cut of that function.

EXAMPLE. Let x -a, where x > 0 and 0 < a < 1, denote the principal value of the indicated power of x; that is, x-a is the positive real number exp(-a lnx). We shall evaluate here the improper real integral

x —a

I -dx (0 < a < 1),

x + 1

which is important in the study of the gamma function.* Note that integral is improper not only because of its upper limit of integration but also because its integrand has an infinite discontinuity at x=0. The integral converges when 0 < a < 1 since the integrand behaves like x—a near x=0 and like x—a—1 as x tends to infinity. We do not, however, need to establish convergence separately for that will be contained in our evaluation of the integral.

We begin by letting Cp and CR denote the circles |z|=p and |z|=R, respectively, where p < 1 < R; and we assign them the orientation We then integrate the branch

Z—a

f(z)=- (|z| > o, 0 < argz < 2n)

z + 1

of the multiple-valued function z—a/(z + 1), with branch cut argz=0, around the simple closed contour indicated. That contour is traced out by a point moving from p to R along the top of the branch cut for f(z), next around CR and back to R, then along the bottom of the cut to p, and finally around Cp back to p R x

Now Q=0 and Q=2n along the upper and lower "edges," respectively, of the cut annulus that is formed. Since

exp(—a logz) exp[—a(lnr + iQ)]

f(z) = reiQ + 1

where z=relQ, it follows that

exp[—a(ln r + i 0)] r

f(z) = r + 1

on the upper edge, where z=rei0, and thatexp[—a(lnr + i2n)] r ae f(z) = r + 1

on the lower edge, where z=rei2n. The residue theorem thus suggests that

fR r a f f—~7dr + f(z) dz- Jp r + 1 JCR Jp= 2ni Res f(z).
z=—1

Our derivation of equation is, of course, only formal since f(z) is not analytic, or even defined, on the branch cut involved. It is, nevertheless, valid and can be fully justified by an argument of this section. The residue in equation can be found by noting that the function

$(z)=z~"=exp(-a logz)=exp[—a(lnr + i9)] (r > 0, 0 < 9 < 2n)

is analytic at z=— 1 and that

$(—1)=exp[—a(ln 1 + in)]=e~ian=0.

This shows that the point z=— 1 is a simple pole of the function and that

Res f(z)=e~ian.

z=—1

Equation can, therefore, be written as

(l-e~lZa7T)[^—dr=2nie~ia7T-[f(z)dz~[f(z)dz.

According to definition of f(z),

Since 0 < a < 1, the values of these two integrals evidently tend to 0 as p and R tend to 0 and to, respectively. Hence, if we let p tend to 0 and then R tend to in equation, we arrive at the result

p to r —a (1 _ £-'2«T) dr=2nie~ia7T,

or

p to r —a e—ian e^an 21

I dr=2ni- ^—• ——= n— :—.

J0 r + 1 1 e—i2an eian eian e—ian

fTO x—a n

I dx=— (0 < a < 1).

Apply the Cauchy-Goursat theorem to the branch

z—a (n 5n)

fz(z)=—— (kl > o, - < argz < — z + 1 \ 2 2

of z—a/(z + 1), integrated around the closed contour on the right

pR r—a^— i2anp f f

- —7—j—df+ f2(z) dz- fZ1z) dz+ f2(z)dz=0. J p r + 1 J Yp J L J yr

Point out why, in the last lines in parts (a) and (b), the branches p(z) and f2(z) of z—a/(z + 1) can be replaced by the branch

z—a

fiz)=——- (kl > 0, 0 < argz < In), z + 1

11.7 DEFINITE INTEGRALS INVOLVING SINES AND COSINES

The method of residues is also useful in evaluating certain definite integrals of the type

/ F(sin9, cos9)d9.

The fact that varies from 0 to 2n leads us to consider 9 as an argument of a point z on a positively oriented circle C centered at the origin. Taking the radius to be unity, we use the parametric representation

$z=e^{i9}$ (0 < 9 < 2n)

$dz/d9=le^{i9}$=iz and that

sin d = e^{i9} — $e^{—i9}$/2i cos 9 = e^{i9} + $e^{—i9}$/2

These relations suggest that we make the substitutions

sinO=2i cos9=2 iz

of a function of z around the circle C. The original integral is, of course, simply a parametric form of integral in accordance with expression When the integrand in integral reduces to a rational function of z, we can evaluate that integral by means of Cauchy's residue theorem once the zeros in the denominator have been located and provided that none lie on C.

I 1 , "A,-. ^=A" o (—1 < a < 1).

This integration formula is clearly valid when a=0, and we exclude that case in our derivation. With substitutions, the integral takes the form

I z' + mU-i dz'

where C is the positively oriented circle |z|=1. The quadratic formula reveals that the denominator of the integrand here has the pure imaginary zeros

/ —1 + Vl — «^{2}\ ./—1 — Vl — «^{2}\ .

So if f(z) denotes the integrand in integral then

$f(z) = (z - z_1)(z - z_2)$

Note that because \a\ < 1,

l + vTr

l-2l=   —   > 1.

Also, since \z1Z2l=1, it follows that $|z_1| < 1$. Hence there are no singular points on C, and the only one interior to it is the point z_1. The corresponding residue B1 is found by writing

S(z) 2/a f(z) = where <p(z) = $Z - z_1 \quad Z - z_2$

This shows that z_1 is a simple pole and that

B\=4>(z\)='

Zl ~ z_2 i*J1 —

Consequently,

Notes

2/a 2n

dz=2niB\ = !c $_{Z2}$ + (2i/a)z — I Vl — a^2

and integration formula follows.

The method just illustrated applies equally well when the arguments of the sine and cosine are integral multiples of Q. One can use equation to write, for example,

el2Q + e-l2Q (eiQ)2 + (eiQ)-2 $_{Z2}$ + Z-2

11.8 ARGUMENT PRINCIPLE

The function f is said to be meromorphic in a domain D if it is analytic throughout D except for poles. Suppose now that f is meromorphic in the domain interior to a positively oriented simple closed contour C and that it is analytic and nonzero on C. The image Y of C under the transformation w=f(z) is a closed contour, not necessarily simple, in the w plane As a point z traverses C in the positive direction, its images w traverses Y in a particular direction that determines the orientation of Y. Note that since f has no zeros on C, the contour Y does not pass through the origin in the w plane.

Let W0 and w be points on Y, where W0 is fixed and is a value of arg w0. Then let arg w vary continuously, starting with the value 00, as the point w begins at the point w0 and traverses Y once in the direction of orientation assigned to it by the mapping w=f(z). When w returns to the point wo, where it started, arg w assumes a particular value of arg wo, which we denote by fai. Thus the change in arg w as w describes T once in its direction of orientation is fa — fao. This change is, of course, independent of the point wo chosen to determine it- Since w=f(z), the number fa — fao is, in fact, the change in argument of f(z) as z describes C once in the positive direction, starting with a point zo; and we write

Ac arg f(z)=fa — fa

The value of AC arg f(z) is evidently an integral multiple of 2n, and the integer

1— Ac arg f(z)

Represents the number of times the point w winds around the origin in the w plane- For that reason, this integer is sometimes called the winding number of T with respect to the origin w=0- It is positive if T winds around the origin in the counterclockwise direction and negative if it winds clockwise around that point- The winding number is always zero when T does not enclose the origin.

The winding number can be determined from the number of zeros and poles of f interior to C. The number of poles is necessarily finite. Likewise, with the understanding that f(z) is not identically equal to zero everywhere else inside C, it is easily that the zeros of f are finite in number and are all of finite order. Suppose now that f has Z zeros and P poles in the domain interior to C. We agree that f has mo zeros at a point zo if it has a zero of order mo there; and if f has a pole of order mp at zo, that pole is to be counted mp times. The following theorem, which is known as the argument principle, states that the winding number is simply the difference Z — P.

Theorem. Let C denote a positively oriented simple closed contour, and suppose that a function f(z) is meromorphic in the domain interior to C;

f(z) is analytic and nonzero on C;

counting multiplicities, Z is the number of zeros and P the number of poles of f(z) inside C.

Then — Ac arg f(z)=Z - P.

To prove this, we evaluate the integral of f '(z)/f(z) around C in two different ways. First, we let z=z(t) (a < t < b) be a parametric representation for C, so that Since, under the transformation w=f(z), the image T of C never passes through the origin in the w plane, the image of any point z=z(t) on C can be expressed in exponential form as w=p(t) exp[i<^(t)]. Thus

f [z(t)]=p(t)e'm (a < t < b);

and, along each of the smooth arcs making up the contour T,

f'[z(t)]z'(t)=^-f[z(t)]=Upit)e^\=p'(t)e"f^ +ip(t)e"f^4>'(t).

Inasmuch as p'(t) and $'(t) are piecewise continuous on the interval a < t < b, we can now use expressions

f'iz) fb p'it) ic But p(b)=p(a) and $(b) - $(a)=Ac arg f(z).

Hence

f dz=iAcargf(z). Jc f(z)

Another way to evaluate integral is to use Cauchy's residue theorem. To be specific, we observe that the integrand f '(z)/f(z) is analytic inside and on C except at the points inside C at which the zeros and poles of f occur. If f has a zero of order m0 at z0.

f(z)=(z - z0)m0g(z), where g(z) is analytic and nonzero at z0. Hence

f '(z0)=m0(z - z0)m0-1g(z) + (z - z0)m0g'(z),

or - m° + g'(z) fiz) z-zo giz)'

Since g'(z)/g(z) is analytic at z0, it has a Taylor series representation about that point; and so equation tells us that f '(z)/f(z) has a simple pole at z0, with residue m0. If, on the other hand, f has a pole of order mp at z0 f(z)=(z - z0)~mp<P(z),

where $(z) is analytic and nonzero at z0. Because expression has the same form as expression, with the positive integer m0 in equation replaced by -mp,it is clear from equation that f '(z)/f(z) has a simple pole at zo, with residue -mp. Applying the residue theorem, then, we find that

[dz=2ni(Z - P). Jc f(z)

The conclusion in the theorem now follows by equating the right-hand sides of equations.

EXAMPLE. The only singularity of the function $1/_{Z2}$ is a pole of order 2 at the origin, and there are no zeros in the finite plane. In particular, this function is analytic and nonzero on the unit circle z=elQ (0 < Q < 2n). If we let C denote that positively oriented circle, our theorem tells us that

AcarB(?)=-2-

That is, the image T of C under the transformation w=1/z^2 winds around the origin w=0 twice in the clockwise direction. This can be verified directly by noting that T has the parametric representation w=e~l2Q(0 < Q < 2n).

11.9 ROUCHE'S THEOREM

The main result in this section is known as Rouche's theorem and is a consequence of the argument principle, It can be useful in locating regions of the complex plane in which a given analytic function has zeros.

Theorem. Let C denote a simple closed contour, and suppose that

two functions f(z) and g(z) are analytic inside and on C;

\f(z)\ > |g(z) | at each point on C.

Then f(z) and f(z) + g(z) have the same number of zeros, counting multiplicities, inside C.

The orientation of C in the statement of the theorem is evidently immaterial. Thus, in the proof here, we may assume that the orientation is positive. We begin with the observation that neither the function f(z) nor the sum f(z) + g(z) has a zero on C, since

\f(z)\> \g(z)|>0 and \f(z) + g(z)\>\\f(z)\-\g(z)\\ > 0

when z is on C.

If Zf and Zf+g denote the number of zeros, counting multiplicities, of f(z) and f(z) + g(z), respectively, inside C,

and this means that under the transformation w=F(z), the image of C lies in the open disk | w - 11 < 1. That image does not, then, enclose the origin w=0. Hence AC arg F(z)=0 and, since equation reduces to Zf+g=Zf, Rouche's theorem is proved.

Notes

EXAMPLE . In order to determine the number of roots of the equation

$z^7 - 4z^3 + z - 1=0$

inside the circle $|z|=1$, write

$f(z)=-4z^3$ and $g(z)=z^7 + z — 1$.

Then observe that $|f(z)|=4|z|^3=4$ and $|g(z)| < |z|^7 + |z| + 1=3$ when $|z|=1$. The conditions in Rouche's theorem are thus satisfied. Consequently, since f(z) has three zeros, counting multiplicities, inside the circle $|z|=1$, so does f(z) + g(z). That is, equation has three roots there.

EXAMPLE Rouche's theorem can be used to give another proof of the fundamental theorem of algebra To give the detals here, we consider a polynomial

$P(z)=a_o + a_1z + a_2z^2 + \quad + a_nz^n \ (a_n=0)$

of degree n (n > 1) and show that it has n zeros, counting multiplicities. We write

$f(z)=a_nz^n, g(z)=a_0 + a_1 z + a_2z^2 + \quad + a_{n-1}z^{n 1}$

and let z be any point on a circle $|z|=R$, where R > 1. When such a point is taken, we see that

$|f(z)|=|a_n|R^n$.

Also,

|g(z)| < |aol + |ai|R + MR2 + ■ ■- + |an-l|Rn-1.

Consequently, since R > 1,

|^(z)I < |a0|Rn-1 + N|Rn—1 + |a2|Rn-1 +• • • + K—1|Rn—^

and it follows that

|g(z)| |a0| + |a1 | + |a2| + "' + |an-1| ,

< < 1

\f(z)\ " MR

if, in addition to being greater than unity,

D |a0| + |a1| + |a2| + '" + |an-1|

R > — •

|an |

That is, lf(z)l > |^(z) | when R > 1 and inequality is satisfied. Rouche's theorem then tells us that f(z) and f(z) + g(z) have the same number of zeros, namely n, inside C. Hence we may conclude that P(z) has precisely n zeros, counting multiplicities, in the plane.

Note how Liouville's theorem in only ensured the existence of at least one zero of a polynomial; but Rouche's theorem actually ensures the existence of n zeros, counting multiplicities.

Check your Progress-1

Discuss Applications Of Residues

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Discuss Argument Principle

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11.9 LET US SUM UP

In this unit we have discussed the definition and example of Applications Of Residues, Evaluation Of Improper Integrals, Improper Integrals From Fourier Analysis, Jordan's Lemma, Integration Along A Branch Cut, Definite Integrals Involving Sines And Cosines, Argument Principle, Rouche's Theorem

11.10 KEYWORDS

Applications Of Residues We turn now to some important applications of the theory of residues, which was developed

Evaluation Of Improper Integrals In calculus, the improper integral of a continuous function f(x) over the semiinfinite interval 0 < x < to is defined by means of the equation

Improper Integrals From Fourier Analysis Residue theory can be useful in evaluating convergent improper integrals

Jordan's Lemma In the evaluation of integrals of the type treated, it is sometimes necessary to use Jordan's lemma * which is stated just below as a theorem.

Integration Along A Branch Cut Cauchy's residue theorem can be useful in evaluating a real integral when part of the path of integration of the function f(z) to which the theorem is applied lies along a branch cut of that function.

Definite Integrals Involving Sines And Cosines The method of residues is also useful in evaluating certain definite integrals

Argument Principle The function f is said to be meromorphic in a domain D if it is analytic throughout D except for poles

Rouche's Theorem The main result in this section is known as Rouche's theorem and is a consequence of the argument principle, It can be useful in locating regions of the complex plane in which a given analytic function has zeros

11.11 QUESTIONS FOR REVIEW

Explain Applications Of Residues

Explain Argument Principle

11.12 ANSWERS TO CHECK YOUR PROGRESS

Applications Of Residues (answer for Check your Progress-1 Q)

Argument Principle (answer for Check your Progress-1 Q)

11.13 REFERENCES

- Introduction To Complex Analysis
- Application Of Complex Analysis
- Variables of Complex Analysis
- Basic of Complex Analysis

UNIT-12 : MAPPING BY ELEMENTARY FUNCTIONS…LINEAR TRANSFORMATIONS

STRUCTURE

12.0 OBJECTIVES

After studying this unit, you should be able to:

Learn, Understand about Mapping By Elementary Functions

Linear Transformations

The Transformation

Linear Fractional Transformations

Square Roots Of Polynomials

Riemann Surfaces

12.1 INTRODUCTION

In this part of the course we will study some basic complex analysis . This is an extremely useful and beautiful part of mathematics and forms the basis of many techniques employed in many branches of mathematic In this section we will study complex functions of a complex variable, Mapping By Elementary Functions, Linear Transformations, The Transformation, Linear Fractional Transformations, Square Roots Of Polynomials, Riemann surfaces.

12.2 MAPPING BY ELEMENTARY FUNCTIONS

The geometric interpretation of a function of a complex variable as a mapping, or transformation the nature of such a function can be displayed graphically, to some extent, by the manner in which it maps certain curves and regions.

In this chapter, we shall see further examples of how various curves and regions are mapped by elementary analytic functions.

LINEAR TRANSFORMATIONS

To study the mapping

w=Az,

where A is a nonzero complex constant and z=0, we write A and z in exponential form:

A=aeia, z=rew. Then w=(ar)ei(a+e),

and we see from equation that transformation expands or contracts the radius vector representing z by the factor a and rotates it through the angle a about the origin. The image of a given region is, therefore, geometrically similar to that region.

Notes

w=z + B,

where B is any complex constant, is a translation by means of the vector representing B. That is, if

w=u + iv, z=x + iy, and B=b\ + ib2, then the image of any point (x,y) in the z plane is the point

(u, v)=(x + b\, y + b2)

in the w plane. Since each point in any given region of the z plane is mapped into the w plane in this manner, the image region is geometrically congruent to the original one.

The general (nonconstant) linear transformation

w=Az + B (A=0)

is a composition of the transformations

Z=Az (A=0) and w=Z + B.

When z=0, it is evidently an expansion or contraction and a rotation, followed by a translation.

EXAMPLE. The mapping

w=(1 + i)z + 2

transforms the rectangular region in the z=(x,y) plane into the rectangular region shown in the w=(u,v) plane there. This is seen by expressing it as a composition of the transformations

Z=(1 + i)z and w=Z + 2.

and z=rexp(i6), one can put the first of transformations in the form

This first transformation thus expands the radius vector for a nonzero point z by the factor \fl and rotates it counterclockwise 7t/4 radians about the origin. The second of transformations is, of course, a translation two units to the right.

12.3 THE TRANSFORMATION

w=1/z

The equation

1

w=- z

establishes a one to one correspondence between the nonzero points of the z and the w planes. Since cT=\z\2, the mapping can be described by means of the successive transformations

The first of these transformations is an inversion with respect to the unit circle |z|=1. That is, the image of a nonzero point z is the point Z with the properties

1

\Z\=— and argZ=argz.

|z|

Thus the points exterior to the circle |z|=1 are mapped onto the nonzero points interior and conversely. Any point on the circle is mapped onto itself. The second of transformations is simply a reflection in the real axis.

If we write transformation as

T(z)=- (z± 0),

z

we can define T at the origin and at the point at infinity so as to be continuous on the extended complex plane.

lim T(z)=oo since lim =lim z=0

W Z-+0 z^o T(z) z^O

and

lim T(z)=0 since lim T (- |=lim z=0.

z^~ z^0 \zj z^0

Notes

In order to make T continuous on the extended plane, then, we write

T(0)=oo, T(oo)=0, and T(z)=-

z

for the remaining values of z. More precisely, the first of limits that the limit

lim T(z)=T(z0),

z^z0

which is clearly true when z0=0 and when z0 =rc>, is also true for those two values of z0. The fact that T is continuous everywhere in the extended plane is now a consequence of limit Because of this continuity, when the point at infinity is involved in any discussion of the function 1/z, we tacitly assume that T (z) is intended.

12.4 LINEAR FRACTIONAL TRANSFORMATIONS

The transformation

w=" — (ad — be f 0), cz + d

where a, b, c, and d are complex constants, is called a linear fractional transformation, or Mobius transformation. Observe that equation can be written in the form

Azw + Bz + Cw + D=0 (AD - BC=0);

and, conversely, any equation of type can be put in the form. Since this alternative form is linear in z and linear in w, another name for a linear fractional transformation is bilinear transformation.

When c=0, the condition ad — bc=0 with equation becomes ad=0; and we see that the transformation reduces to a nonconstant linear function. When c=0, equation can be written

a bc ad 1

w=—| (ad — bc f 0). c c cz + d

So, once again, the condition ad — bc=0 ensures that we do not have a constant function. The transformation w=1 /z is evidently a special case of transformation when c=0.

Equation reveals that when c=0, a linear fractional transformation is a composition of the mappings.

1 a bc ad

Z=cz + d, W=—, w=—| W (ad — bc f 0). Z c c

It thus follows that, regardless of whether c is zero or nonzero, any linear fractional transformation transforms circles and lines into circles and lines because these special linear fractional transformations do Solving equation for z, we find that

dw b z = (ad — bc f 0).

When a given point w is the image of some point z under transformation the point z is retrieved by means of equation. If c=0, so that a and d are both nonzero, each point in the w plane is evidently the image of one and only one point in the z plane. The same is true if c=0, except when w=a/c since the denominator in equation vanishes if w has that value. We can, however, enlarge the domain of definition of transformation in order to define a linear fractional transformation T on the extended z plane such that the point w=a/c is the image of z=ro when c=0. We first write

T(z)=~ — (ad — be f 0). cz + d

We then write

T (<x>)=ro if c=0 and T(oo)=- and Ty—j=oo if c f 0.

When its domain of definition is enlarged in this way, the linear fractional transformation is a one to one mapping of the extended z plane onto the extended w plane. That is, T(z\)=T(z_2) whenever z_1=z_2; and, for each point w in the second plane, there is a point z in the first one such that T(z)=w. Hence, associated with the transformation T, there is an inverse transformation T-1, which is defined on the extended w plane as follows:

Notes

T-1(w)=z if and only if T(z)=w.

From equation, we see that

1 —dw+b

T (w)= (ad — be f 0).

cw — a

Evidently, T-1 is itself a linear fractional transformation, where

T—1(^) =ro if c=0 and

r_1(-)=oo and T~l(oo) =-- if cf 0.

If T and S are two linear fractional transformations, then so is the composition S[T(z)]. This can be verified by combining expressions of the type. Note that, in particular, T—1 [T(z)]=z for each point z in the extended plane.

There is always a linear fractional transformation that maps three given distinct points z_1, z_2, and Z3 onto three specified distinct points wi, W2, and W3, respectively. Verification of this will appear in Sec. 94, where the image w of a point z under such a transformation is given implicitly in terms of z. We illustrate here a more direct approach to finding the desired transformation.

EXAMPLE . Let us find the special case of transformation that maps the points

z_1=-1, z_2=0, and Z3=1

onto the points

wi=—i, W2=1, and W3=i.

Since 1 is the image of 0, expression tells us that 1=b/d, or d=b. Thus

w=———-r [b(a-c)^ 0], cz + b

Then, since —1 and 1 are transformed into —i and i, respectively, it follows that

ic — ib=—a + b and ic + ib=a + b.

Adding corresponding sides of these equations, we find that c=—ib ; and subtrac- tion reveals that a=ib. Consequently,

ibz + b b(iz + 1)

—ibz+b b(—iz + 1) '

We can cancel out the nonzero number b in this last fraction and write

iz + 1 w=—: 7- —iz + 1

This is, of course, the same as u> = i + z

which is obtained by assigning the value i to the arbitrary number b.

EXAMPLE. Suppose that the points

z_1=1, z_2=0, and z3=-1

are to be mapped onto

W1=i, W2 =<x>, and W3=1.

Since W2=ro corresponds to z_2=0, c=0 and d=0 in equation Hence

w=~ (be ^ 0).

cz

Then, because 1 is to be mapped onto i and —1 onto 1, we have the relations

ic=a + b, —c=—a + b ;

and it follows that

2a=(1 + i)c, 2b=(i — 1)c.

Finally, if we multiply numerator and denominator in the quotient by 2, make these substitutions for 2a and 2b, and then cancel out the nonzero number c, we arrive at

(i + 1)z + (i — 1)

(14) w = 2z

AN IMPLICIT FORM

Notes

The equation

(w - U'l)(U>2 - U'3) _ (z -Zl)(ZZ -Z3)

(W - W3)(W2 ~ Wl) (Z ~ Z3)(z_2 -til) defines (implicitly) a linear fractional transformation that maps distinct points z_1, z_2, and Z3 in the finite z plane onto distinct points w1, W2, and W3, respectively, in the finite w plane.6 To verify this, we write equation as

(z — Z3)(w — W1)(z_2 — z_1)(W2 — W3)=(z — Z\)(w — W3)(z_2 — Z3)(w2 — W1).

If z=z_1, the right-hand side of equation is zero; and it follows that w=w1. Similarly, if z=z3, the left-hand side is zero and, consequently, w=W3. If z=z_2, we have the linear equation

(w — w1)(w2 — w3)=(w — w3)(w2 — w1),

whose unique solution is w=w2. One can see that the mapping defined by equation is actually a linear fractional transformation by expanding the products in equation and writing the result in the form

Azw + Bz + Cw + D=0.

The condition AD — BC=0, which is needed with equation is clearly satisfied since, as just demonstrated, equation does not define a constant function fractional transformation mapping the points z_1, z_2, and z3 onto w1, w2, and w3, respectively.

EXAMPLE. The transformation found in Example required that

z_1=— 1, z_2=0, Z3=1 and W1=—i, W2=1, W3=i.

Using equation to write

(w + i)(l - i) _ (z + 1)(0 - 1)

(w - 2)(1 + 2) (z - 1)(0 + 1) and then solving for w in terms of z, we arrive at the transformation i — z

w = ,

i + z

found earlier.

If equation is modified properly, it can also be used when the point at infinity is one of the prescribed points in either the (extended) z or w plane. Suppose, for instance, that z_1=to. Since any linear fractional transformation is continuous on the extended plane, we need only replace z_1 on the right-hand side of equation by 1/z_1, clear fractions, and let z_1 tend to zero:

(z — 1/z_1)(z_2 — z3) z_1(z1z — 1)(z_2 — z3) z_2 — z3

lim • —=lim = .

z_1^0 (z — z3)(z_2 — 1/z_1) z_1 z_1^0 (z — z3)(z1z2 — 1) z — z3

The desired modification of equation is, then,

(W - W\)(W2 ~ U'3) _ z_2 ~ Z3 (W - W3)(W2 ~ W\) Z~Z3~

Note that this modification is obtained formally by simply deleting the factors involving z_1 in equation . It is easy to check that the same formal approach applies when any of the other prescribed points is to.

EXAMPLE The prescribed points were z_1=1, z_2=0, z3=—1 and W1=i, W2=to, W3=1.

In this case, we use the modification

w - W\ _ (Z - Zl)(z_2 - Z3)

W - U'3 (z - Z3)(z_2 - z_1) of equation (1), which tells us that

w - i (z — 1)(0 + 1) w - 1=U + 1)(0 — 1)'

Solving here for w, we have the transformation obtained earlier:

(i + 1)z + (i — 1) w 2z

12.5 SQUARE ROOTS OF POLYNOMIALS

We now consider some mappings that are compositions of polynomials and square roots.

EXAMPLE. Branches of the double-valued function (z — z0)1/2 can be obtained by noting that it is a composition of the translation Z=z — z0 with the double-valued function z_1/2. Each branch of z_1/2 yields a branch of (z — zo)1/2. More precisely, when Z=Rel6, branches of z_1/2 are

z_1/2=V/?exp — (R > 0,a < 9 < a + 2jt), according to equation (8) in Sec. 97. Hence if we write

R=|z — Z01, θ=Arg (z — Z0), and 6=arg(z — Z0), two branches of (z — z0)1/2 are

Go(z)=VrR exp(R > 0, — n < θ < n) and i0

g0(z)=</Rexp — (R > 0, 0 < 6 < 2n).

The branch of z_1/2 that was used in writing G0(z) is defined at all points in the Z plane except for the origin and points on the ray Arg Z=n. The transformation w=G0(z) is, therefore, a one to one mapping of the domain

|z — z01 > 0, —n< Arg (z — z0) < n

onto the right half Re w > 0 of the w plane The transformation w=g0(z) maps the domain

|z — z01 > 0, 0 < arg(z — z0) < 2n in a one to one manner onto the upper half plane Im w > 0

12.6 RIEMANN SURFACES

The remaining two sections of this chapter constitute a brief introduction to the concept of a mapping defined on a Riemann surface, which is a generalization of the complex plane consisting of more than one sheet. The theory rests on the fact that at each point on such a surface only one value of a given multiple-valued function is assigned. The material in these two sections will not be used in the chapters to follow, and the reader may skip to without disruption. Once a Riemann surface is devised for a given function, the function is single valued on the surface and the theory of single-valued functions applies there. Complexities arising because the function is multiple-valued are thus relieved by a geometric device. However, the description of those surfaces and the arrangement of proper connections between the sheets can become quite involved. We limit our attention to fairly simple examples and begin with a surface for log z.

EXAMPLE. Corresponding to each nonzero number z, the multiple valued function

log z=ln r + iO

has infinitely many values. To describe log z as a single-valued function, we replace the z plane, with the origin deleted, by a surface on which a new point is located whenever the argument of the number z is increased or decreased by 2n, or an integral multiple of 2n.

We treat the z plane, with the origin deleted, as a thin sheet R0 which is cut along the positive half of the real axis. On that sheet, let Q range from 0 to 2n. Let a second sheet R1 be cut in the same way and placed in front of the sheet R0 . The lower edge of the slit in R0 is then joined to the upper edge of the slit in R1. On R1, the angle Q ranges from 2n to 4n; so, when z is represented by a point on R1, the imaginary component of log z ranges from 2n to 4n.

A sheet R2 is then cut in the same way and placed in front of R1. The lower edge of the slit in R1 is joined to the upper edge of the slit in this new sheet, and similarly for sheets R3, R4,... .A sheet R—1 on which Q varies from 0 to —2n is cut and placed behind R0, with the lower edge of its slit connected to the upper edge of the slit in R0; the sheets R—2, R—3,... are constructed in like manner. The coordinates r and Q of a point on any sheet can be considered as polar coordinates of the projection of the point onto the original z plane, the angular coordinate Q being restricted to a definite range of 2n radians on each sheet.

Consider any continuous curve on this connected surface of infinitely many sheets. As a point z describes that curve, the values of log z vary continuously since Q, in addition to r, varies continuously; and log z now assumes just one value corresponding to each point on the curve. For example, as the point makes a complete cycle around the origin on the sheet R0 over the path indicated the angle changes from 0 to 2n. As it moves across the ray Q=2n, the point passes to the sheet R1 of the surface. As the point completes a cycle in R1, the angle Q varies from 2n to 4n; and as it crosses the ray Q=4n, the point passes to the sheet R2.

The surface described here is a Riemann surface for logz. It is a connected surface of infinitely many sheets, arranged so that log z is a single-valued function of points on it.

The transformation w=log z maps the whole Riemann surface in a one to one manner onto the entire w plane. The image of the sheet R0 is the strip $0 < v < 2n$ As a point z moves onto the sheet R1 its image w moves upward across the line v=2n, as indicated.

Note that logz, defined on the sheet R1, represents the analytic continuation of the single-valued analytic function

f(z)=ln r + id $(0 < d < 2n)$

upward across the positive real axis. In this sense, log z is not only a single-valued function of all points z on the Riemann surface but also an analytic function at all points there.

The sheets could, of course, be cut along the negative real axis or along any other ray from the origin, and properly joined along the slits, to form other Riemann surfaces for log z.

EXAMPLE . Corresponding to each point in the z plane other than the origin, the square root function

z1'2=2

has two values. A Riemann surface for z_1/2 is obtained by replacing the z plane with a surface made up of two sheets R0 and R,, each cut along the positive real axis and with R, placed in front of R0. The lower edge of the slit in R0 is joined to the upper edge of the slit in R,, and the lower edge of the slit in R, is joined to the upper edge of the slit in R0.

As a point z starts from the upper edge of the slit in R0 and describes a continuous circuit around the origin in the counterclockwise direction the angle 0 increases from 0 to 2n. The point then passes from the sheet Ro to the sheet Ri, where 0 increases from 2n to 4n. As the point moves still further, it passes back to the sheet R0, where the values of 0 can vary from 4n to 6n or from 0 to 2n, a choice that does not affect the value of $_{Z1}$/2, etc. Note that the value of $_{Z1}$/2 at a point where the circuit passes from the sheet R0 to the sheet R1 is different from the value of $_{Z1}$/2 at a point where the circuit passes from the sheet R1 to the sheet R0.

We have thus constructed a Riemann surface on which $_{Z1}$/2 is single-valued for each nonzero z. In that construction, the edges of the sheets R0 and R1 are joined in pairs in such a way that the resulting surface is closed and connected. The points where two of the edges are joined are distinct from the points where the other two edges are joined. Thus it is physically impossible to build a model of that Riemann surface. In visualiZ1ng a Riemann surface, it is important to understand how we are to proceed when we arrive at an edge of a slit.

The origin is a special point on this Riemann surface. It is common to both sheets, and a curve around the origin on the surface must wind around it twice in order to be a closed curve. A point of this kind on a Riemann surface is called a branch point.

The image of the sheet R0 under the transformation w=$_{Z1}$/2 is the upper half of the w plane since the argument of w is 0/2 on R0, where 0 < 0/2 < n. Likewise, the image of the sheet R1 is the lower half of the w plane. As defined on either sheet, the function is the analytic continuation, across the cut, of the function defined on the other sheet. In this respect, the single-valued function $_{Z1}$/2 of points on the Riemann surface is analytic at all points except the origin.

Check your Progress-1

Discuss Mapping By Elementary Functions

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Discuss Riemann Surfaces

12.7 LET US SUM UP

In this unit we have discussed the definition and example of Mapping By Elementary Functions, Linear Transformations, The Transformation, Linear Fractional Transformations, Square Roots Of Polynomials, Riemann Surfaces

12.8 KEYWORDS

Mapping by Elementary Functions Complex Analysis, Basic of Complex Analysis, Complex Functions & Variables, Complex Variables, Introduction To Complex Analysis, Application Of Complex Analysis & Variables, Complex Functions**,** Complex Numbers & Analysis, The Complex Number System

Linear Transformations The geometric interpretation of a function of a complex variable as a mapping, or transformation the nature of such a function can be displayed graphically, to some extent, by the manner in which it maps certain curves and regions.

The Transformation w=1/z The equation

Linear Fractional Transformations The transformation w=" — (ad — be f 0),

Square Roots Of Polynomials We consider some mappings that are compositions of polynomials and square roots.

Riemann Surfaces The remaining two sections of this chapter constitute a brief introduction to the concept of a mapping defined on a Riemann surface, which is a generalization of the complex plane consisting of more than one sheet.

12.9 QUESTIONS FOR REVIEW

Explain Mapping by Elementary Functions

Explain Riemann Surfaces

12.10 ANSWERS TO CHECK YOUR PROGRESS

Mapping by Elementary Functions (answer for Check your Progress-1 Q)

Riemann Surfaces (answer for Check your Progress-1 Q)

12.11 REFERENCES

- Complex Numbers & Analysis
- The Complex Number System
- Complex Analysis
- Complex Variables
- Application Of Complex Analysis & Variables

Notes

UNIT-13 : CONFORMAL MAPPING

STRUCTURE

13.0 OBJECTIVES

After studying this unit, you should be able to:

Learn, Understand about Conformal Mapping

Preservation Of Angles

Scale Factors

Local Inverses

Harmonic Conjugates

Transformations Of Harmonic Functions

Transformations Of Boundary Conditions

Two-Dimensional Fluid Flow

The Stream Function

Flows Around A Corner And Around A Cylinder

13.1 INTRODUCTION

In this part of the course we will study some basic complex analysis . This is an extremely useful and beautiful part of mathematics and forms the basis of many techniques employed in many branches of mathematic In this section we will study complex functions of a complex variable, Conformal Mapping, Preservation Of Angles, Scale Factors, Local Inverses, Harmonic Conjugates, Transformations Of Harmonic Functions, Transformations Of Boundary Conditions, Two-Dimensional Fluid Flow, The Stream Function, Flows Around A Corner And Around A Cylinder

13.2 CONFORMAL MAPPING

In this chapter, we introduce and develop the concept of a conformal mapping, with emphasis on connections between such mappings and harmonic functions

13.3 PRESERVATION OF ANGLES

Let C be a smooth arc represented by the equation

z=z(t) (a < t < b), and let f(z) be a function defined at all points z on C. The equation

w=f [z(t)] (a < t < b)

is a parametric representation of the image Y of C under the transformation w=f(z).

Suppose that C passes through a point z0=z(t0) (a < t0 < b) at which f is analytic and that f '(z0)=0. According to the chain rule verified

If w(t)=f [z(t)], then

w'(t0)=f '[z(t0)]z'(t0); and this means that

arg w'(t0)=arg f '[zt)] + arg z'(t0).

Statement is useful in relating the directions of C and Y at the points zo and wo=f(za), respectively.

To be specific, let 60 denote a value of argz'(t0) and let $0 be a value of arg w'(t0). According to the discussion of unit tangent vectors T near the end of the number 60 is the angle of inclination of a directed line tangent to C at Z0 and 60 is the angle of inclination of a directed line tangent to Y at the point W0=f(z0). In view of statement, there is a value ^0 of arg f '[z(t0)] such that $0=f0 + 60.

Thus $0 — 60=^0, and we find that the angles $0 and 60 differ by the angle of rotation

f0=arg f '(z0).

Now let C1 and C2 be two smooth arcs passing through z0 , and let 61 and 62 be angles of inclination of directed lines tangent to C1 and C2 , respectively, at z0 .   We know from the preceding paragraph that the quantities

$1=f0 + 61 and $2=f0 + 62

are angles of inclination of directed lines tangent to the image curves Y1 and Y 2, respectively, at the point W0=f(z0). Thus $2 — $1=62 — 61; that is, the angle $2 — $1 from Y1 to Y2 is the same in magnitude and sense as the angle 62 — 61 from C1 to C2.

Because of this angle-preserving property, a transformation w=f(z) is said to be conformal at a point Z0 if f is analytic there and f '(z0)=0. Such a transformation is actually conformal at each point in some neighborhood of Z0- For it must be analytic in a neighborhood of Z0 and since its derivative f' is continuous in that neighborhood is also a neighborhood of Z0 throughout which f '(z)=0.

A transformation w=f(z), defined on a domain D, is referred to as a conformal transformation, or conformal mapping, when it is conformal at each point in D. That is, the mapping is conformal in D if f is analytic in D and its derivative f' has no zeros there. Each of the elementary functions can be used to define a transformation that is conformal in some domain.

EXAMPLE. The mapping w=ez is conformal throughout the entire z plane since (ez)'=ez=0 for each z. Consider any two lines x=C1 and y=C2 in the z plane, the first directed upward and the second directed to the right. According to their images under the mapping w=ez are a positively oriented circle centered at the origin and a ray from the origin, respectively is a right angle in the negative direction, and the same is true of the angle between the circle and the ray at the corresponding point in the w plane. The conformality of the mapping w=ez is also illustrated

EXAMPLE. Consider two smooth arcs which are level curves u(x,y)=C1 and v(x,y)=C2 of the real and imaginary components, respectively, of a function

f(z)=u(x, y) + iv(x, y),

and suppose that they intersect at a point z0 where f is analytic and f '(z0)=0. The transformation w=f(z) is conformal at z0 and maps these arcs into the lines u=C1 and v=C2, which are orthogonal at the point

$w_0=f(z_0)$. According to our theory, then, the arcs must be orthogonal at z_0. This has already been verified

A mapping that preserves the magnitude of the angle between two smooth arcs but not necessarily the sense is called an isogonal mapping.

EXAMPLE . The transformation w=z, which is a reflection in the real axis, is isogonal but not conformal. If it is followed by a conformal transformation, the resulting transformation w=f(z) is also isogonal but not conformal.

Suppose that f is not a constant function and is analytic at a point z_0 . If, in addition, $f'(z_0)=0$, then z_0 is called a critical point of the transformation

w=f(z).

EXAMPLE . The point zo=0 is a critical point of the transformation

$w=1 + z^2$,

which is a composition of the mappings

$Z=z^2$ and $w=1 + Z$.

A ray Q=a from the point $Z_0=0$ is evidently mapped onto the ray from the point $W_0=1$ whose angle of inclination is 2a, and the angle between any two rays drawn from $z_0=0$ is doubled by the transformation.

More generally, it can be shown that if z_0 is a critical point of a transformation w=f(z), there is an integer m (m > 2) such that the angle between any two smooth arcs passing through z_0 is multiplied by m under that transformation. The integer m is the smallest positive integer such that $f^{(m)}(z_o)=0$. Verification of these facts is left to the exercises.

13.4 SCALE FACTORS

Another property of a transformation w=f(z) that is conformal at a point z_0 is obtained by considering the modulus of $f'(z_o)$. From the definition of derivative and a property of limits involving moduli that was derived

Notes

Now |z — zo | is the length of a line segment joining zo and z, and If(z) — f (zo) | is the length of the line segment joining the points f(zo) and f(z) in the w plane. Evidently, then, if z is near the point z0 , the ratio

l/U)-/Uo)l |z -zol

of the two lengths is approximately the number |f'(zo)|. Note that |f '(zo)| represents an expansion if it is greater than unity and a contraction if it is less than unity.

Although the angle of rotation arg f' (z) and the scale factor If '(z)| vary, in general, from point to point, it follows from the continuity of f' that their values are approximately arg f '(zo) and |f '(zo)| at points z near zo. Hence the image of a small region in a neighborhood of zo conforms to the original region in the sense that it has approximately the same shape. A large region may, however, be transformed into a region that bears no resemblance to the original one.

EXAMPLE. When f(z)=$_{Z2}$, the transformation

w=f(z)=x2 — y2 + i2xy

is conformal at the point z=1 + i, where the half lines

y=x (x > 0) and x=1 (y > 0)

intersect. We denote those half lines by Ci and C2 with positive sense upward. Observe that the angle from C1 to C2 is n/4 at their point of intersection. Since the image of a point z=(x,y) is a point in the w plane whose rectangular coordinates are

u=x2 — y2 and v=2xy, the half line C1 is transformed into the curve Y1 with parametric representation

u=0, v=2x2 (0 < x < to).

Thus Y1 is the upper half v > 0 of the v axis. The half line C2 is transformed into the curve Y2 represented by the equations

u=1 — y2, v=2y (0 < y < to).

Hence Y2 is the upper half of the parabola v2=—4(u — 1). Note that in each case, the positive sense of the image curve is upward.

If u and v are the variables in representation for the image curve Y2 , then

dv dv/dy 2 2 du du/dy —2y v

In particular, dv/du=—1 when v=2. Consequently, the angle from the image curve Y1 to the image curve Y2 at the point w=f(1 + i)=2i is n/4, as required by the conformality of the mapping at z=1 + i. The angle of rotation n/4 at the point z=1 + i is, of course, a value of

arg[/ (1 +2)]=arg[2(l +/)]=— + 2nn (n=0, ±1, ±2).

The scale factor at that point is the number

|/'(1 + /)|=|2(1 + /)|=2V2.

To illustrate how the angle of rotation and the scale factor can change from point to point, we note that they are 0 and 2, respectively, at the point z=1 since f '(1)=2. where the curves C2 and T2 are the ones just discussed and where the nonnegative x axis C3 is transformed into the nonnegative u axis T3.

13.5 LOCAL INVERSES

A transformation w=f(z) that is conformal at a point Z0 has a local inverse there. That is, if W0=f(z0), then there exists a unique transformation z=g(w), which is defined and analytic in a neighborhood N of W0, such that g(w0)=z0 and f [g(w)]=w for all points w in N. The derivative of g(w) is, moreover,

g'(w) = .

W f'(z)

We note from expression that the transformation z=g(w) is itself conformal at w0.

Assuming that w=f(z) is, in fact, conformal at z0 , let us verify the existence of such an inverse, which is a direct consequence of results in

Notes

advanced calculus. the conformality of the transformation $w=f(z)$ at z_0 implies that there is some neighborhood of z_0 throughout which f is analytic. Hence if we write

$z=x + iy$, $z_0=X_0 + iy_0$, and $f(z)=u(x,y) + iv(x,y)$,

we know that there is a neighborhood of the point (x_0, y_0) throughout which the functions $u(x,y)$ and $v(x,y)$, along with their partial derivatives of all orders, are continuous

Now the pair of equations

$u=u(x,y)$, $v=v(x,y)$

represents a transformation from the neighborhood just mentioned into the uv plane. Moreover, the determinant

which is known as the Jacobian of the transformation, is nonzero at the point (x_0, y_0). For, in view of the Cauchy-Riemann equations $u_x=v_y$ and $u_y=-v_x$, one can write J as

$J=(u_x)^2 + (v_x)^2=|f'(z)|^2$;

and $f'(z_0)=0$ since the transformation $w=f(z)$ is conformal at z_0. The above continuity conditions on the functions $u(x,y)$ and $v(x,y)$ and their derivatives, together with this condition on the Jacobian, are sufficient to ensure the existence of a local inverse of transformation at (x_0, y_o). That is, if

$U_0=u(x_0, y_0)$ and $V_0=v(x_0, y_0)$, then there is a unique continuous transformation

$x=x(u, v)$, $y=y(u, v)$,

defined on a neighborhood N of the point (u_0, V_0) and mapping that point onto (x_0, y_0), such that equations hold when equations hold. Also, in addition to being continuous, the functions have continuous first-order partial derivatives satisfying the equations

ra 1111 xu=jVy, Xv=-J»y=~Jvx< yv=J"x

throughout N.

If we write $w=u + iv$ and $W_0=u_0 + iv_0$, as well as

$g(w)=x(u, v) + iy(u, v)$,

the transformation $z=g(w)$ is evidently the local inverse of the original transformation $w=f(z)$ at z_0. Transformations can be written

$u + iv=u(x, y) + iv(x, y)$ and $x + iy=x(u, v) + iy(u, v)$;

and these last two equations are the same as

$w=f(z)$ and $z=g(w)$,

where g has the desired properties. Equations can be used to show that g is analytic in N. Details are left to the exercises, where expression for $g'(w)$ is also derived.

EXAMPLE If $f(z)=ez$, the transformation $w=f(z)$ is conformal everywhere in the z plane and, in particular, at the point $z_0=2ni$. The image of this choice of z_0 is the point $w_0=1$. When points in the w plane are expressed in the form $w=p \exp(i\wedge)$, the local inverse at z_0 can be obtained by writing $g(w)=\log w$, where $\log w$ denotes the branch

logw=lnp + i$ (p > 0,n<6< 3n)

of the logarithmic function, restricted to any neighborhood of w_0 that does not contain the origin. Observe that

$g(1)=\ln 1 + i\, 2n=2ni$

and that when w is in the neighborhood,

$f[g(w)]=\exp(\log w)=w$.

Also

^ d ^ 1 1 g (W)=— logw=— = dw   w exp z

in accordance with equation

Note that if the point $Z_0=0$ is chosen, one can use the principal branch

Log w=ln p + (p > 0, — n < <p < n)

of the logarithmic function to define g. In this case, $g(1)=0$.

Notes

13.6 HARMONIC CONJUGATES

If a function

$f(z)=u(x, y) + iv(x, y)$

is analytic in a domain D, then the real-valued functions u and v are harmonic in that domain. That is, they have continuous partial derivatives of the first and second order in D and satisfy Laplace's equation there:

UXX + Uyy=0, Vxx + Vyy=0.

We had seen earlier that the first-order partial derivatives of u and v satisfy the Cauchy-Riemann equations

Ux=Vy, Uy=Vx ;

and, as pointed out in Sec. 26, v is called a harmonic conjugate of u.

Suppose now that u(x,y) is any given harmonic function defined on a simply connected domain D. In this section, we show that u(x, y) always has a harmonic conjugate v(x,y) in D by deriving an expression for v(x,y).

To accomplish this, we first recall some important facts about line integrals in advanced calculus.7 Suppose that P(x,y) and Q(x,y) have continuous first-order partial derivatives in a simply connected domain D of the xy plane, and let (x0, y0) and (x,y) be any two points in D. If Py=Qx everywhere in D, then the line integral

J P(s,t)ds + Q(s,t)dt

from (x0, y0) to (x,y) is independent of the contour C that is taken as long as the contour lies entirely in D. Furthermore, when the point (x0, y0) is kept fixed and (x, y) is allowed to vary throughout D, the integral represents a single-valued function

r (x,y) F(x,y)=I P(s,t)ds + Q(s,t)dt J(x0,y0) of x and y whose first-order partial derivatives are given by the equations

Fx(x,y)=P(x,y), Fy(x,y)=Q(x,y).

Note that the value of F is changed by an additive constant when a different starting point (x0, y0) is taken.

Returning to the given harmonic function u(x,y), observe how it follows from Laplace's equation uxx + uyy=0 that

(-uy)y=(ux)x

everywhere in D. Also, the second-order partial derivatives of u are continuous in D; and this means that the first-order partial derivatives of -uy and ux are continuous there. Thus, if (x0, y0) is a fixed point in D, the function

r (x,y) v(x,y)=I -ut(s,t)ds + us(s,t)dt (x0,y0)

is well defined for all (x,y) in D; and, according to equations,

Vx(x,y)=—uy(x, y), Vy(x,y)=ux(x,y).These are the Cauchy-Riemann equations. Since the first-order partial derivatives of u are continuous, it is evident from equations that those derivatives of v are also continuous. Hence u(x, y) + iv(x, y) is an analytic function in D; and v is, therefore, a harmonic conjugate of u.

The function v defined by equation is, of course, not the only harmonic conjugate of u. The function v(x, y) + c, where c is any real constant, is also a harmonic conjugate of u.

EXAMPLE. Consider the function u(x,y)=xy, which is harmonic through out the entire xy plane. According to equation, the function

r (x,y) v(x,y)=I -sds + t dt J (0,0)

is a harmonic conjugate of u(x, y). The integral here is readily evaluated by inspection. It can also be evaluated by integrating first along the horizontal path from the point (0, 0) to the point (x, 0) and then along the vertical path from (x, 0) to the point (x, y). The result is

1 2 1 2

Notes

v(x,y)=--U+-y\ and the corresponding analytic function is

f(z)=xy-l-(x2 -y2)=-l-Z2.

13.7 TRANSFORMATIONS OF HARMONIC FUNCTIONS

The problem of finding a function that is harmonic in a specified domain and satisfies prescribed conditions on the boundary of the domain is prominent in applied mathematics. If the values of the function are prescribed along the boundary, the problem is known as a boundary value problem of the first kind, or a Dirichlet problem. If the values of the normal derivative of the function are prescribed on the boundary, the boundary value problem is one of the second kind, or a Neumann problem. Modifications and combinations of those types of boundary conditions also arise.

The domains most frequently encountered in the applications are simply connected; and, since a function that is harmonic in a simply connected domain always has a harmonic conjugate solutions of boundary value problems for such domains are the real or imaginary components of analytic functions.

EXAMPLE The function

T(x,y)=e-y sinx

satisfies a certain Dirichlet problem for the strip 0 < x < n,y > 0 and noted that it represents a solution of a temperature problem. The function T(x,y), which is actually harmonic throughout the xy plane, is the real component of the entire function

-ielz=e—y sinx — ie—y cosx.

It is also the imaginary component of the entire function eiz.

Sometimes a solution of a given boundary value problem can be discovered by identifying it as the real or imaginary component of an analytic function. But the success of that procedure depends on the simplicity of the problem and on one's familiarity with the real and imaginary components of a variety of analytic functions. The following theorem is an important aid.

Theorem. Suppose that an analytic function

w=f(z)=u(x,y) + iv(x, y)

maps a domain Dz in the z plane onto a domain Dw in the w plane. If h(u, v) is a harmonic function defined on Dw, then the function

H(x,y)=h[u(x,y),v(x,y)] is harmonic in Dz.

We first prove the theorem for the case in which the domain Dw is simply connected. that property of Dw ensures that the given harmonic function h(u,v) has a harmonic conjugate g(u,v). Hence the function

<h(w)=h(u, v) + ig(u, v)

is analytic in Dw. Since the function f(z) is analytic in Dz, the composite function &[f(z)] is also analytic in Dz. Consequently, the real part h[u(x,y),v(x,y)] of this composition is harmonic in Dz.

If Dw is not simply connected, we observe that each point w0 in Dw has a neighborhood |w — w0| < e lying entirely in Dw. Since that neighborhood is simply connected, a function of the type is analytic in it. Furthermore, since f is continuous at a point z0 in Dz whose image is w0 , there is a neighborhood $\left|Z - Z_0\right|$ <8 whose image is contained in the neighborhood | w — w0| < e. Hence it follows that the composition <h[f (z)] is analytic in the neighborhood $\left|Z - Z_0\right|$ < 8, and we may conclude that h[u(x, y), v(x, y)] is harmonic there. Finally, since w0 was arbitrarily chosen in Dw and since each point in Dz is mapped onto such a point under the transformation w=f(z), the function h[u(x, y), v(x, y)] must be harmonic throughout Dz.

The proof of the theorem for the general case in which Dw is not necessarily simply connected can also be accomplished directly by means of the chain rule for partial derivatives. The computations are, however, somewhat involved

Notes

EXAMPLE . The function h(u,v)=e—v sinu is harmonic in the domain Dw consisting of all points in the upper half plane v > 0 transformation is w=z2, we have u(x, y)=x2 — y2 and v(x, y)=2xy; moreover, the domain Dz consisting of the points in the first quadrant x > 0, y > 0 of the z plane is mapped onto the domain Dw,

function

H(x, y)=e—2xy sin(x2 — y2)

is harmonic in Dz.

EXAMPLE. A minor modification that as a point z=r exp(iθ0) (—n/2 < θ0 < n/2) travels outward from the origin along a ray θ=θ0 in the z plane, its image under the transformation

w=Log z=lnr + iθ (r > 0, — n < θ < n)

travels along the entire length of the horizontal line v=θ0 in the w plane. So the right half plane x > 0 is mapped onto the horizontal strip —n/2 < v < n/2. By considering the function

h(u, v)=Im w=v, which is harmonic in the strip, and writing

Log z=In \/x2 + v2 + z'arctan —, x

where —n/2 < arctan t < n/2, we find that

H(x, y)=arctan — is harmonic in the half plane x > 0.

13.8 TRANSFORMATIONS OF BOUNDARY CONDITIONS

The conditions that a function or its normal derivative have prescribed values along the boundary of a domain in which it is harmonic are the most common, although not the only, important types of boundary conditions. In this section, we show that certain of these conditions remain unaltered under the change of variables associated with a conformal transformation A boundary value problems. The basic technique there is to transform a given boundary value problem in the xy plane into a simpler one in the uv plane and then to use the theorems of this to write the solution of the original problem in terms of the solution obtained for the simpler one.

Theorem. Suppose that a transformation w=f(z)=u(x, y) + iv(x,y)

is conformal on a smooth arc C, and let T be the image of C under that transformation. If a function h(u, v) satisfies either of the conditions

dh h=ho or —=0 dn

along T , where ho is a real constant and dh/dn denotes derivatives normal to T, then the function

H(x,y)=h[u(x,y),v(x,y)] satisfies the corresponding condition

along C, where dH/dN denotes derivatives normal to C.

To show that the condition h=ho on T implies that H=ho on C, we note from equation that the value of H at any point (x,y) on C is the same as the value of h at the image (u, v) of (x, y) under transformation point (u, v) lies on T and since h=h0 along that curve, it follows that H=h0 along C.

Suppose, on the other hand, that dh/dn=0 on T. From calculus, we know

that

dh —=(grad h) ■ n, dn

where grad h denotes the gradient of h at a point (u,v) on T and n is a unit vector normal to T at (u, v). Since dh/dn=0 at (u, v), equation tells us that grad h is orthogonal to n at (u,v). That is, grad h is tangent to T there But gradients are orthogonal to level curves; and, because grad h is tangent to T, we see that T is orthogonal to a level curve h(u, v)=c passing through (u, v). v

grad h h(u, v)=c

Now, according to equation, the level curve H(x,y)=c in the z plane can be written

h[u(x, y), v(x, y)]=c ;

Notes

and so it is evidently transformed into the level curve h(u,v)=c under transformation. Furthermore, since C is transformed into T and T is orthogonal to the level curve h(u,v)=c, as demonstrated in the preceding paragraph, it follows from the conformality of transformation that C is orthogonal to the level curve H(x, y)=c at the point (x, y) corresponding to (u, v). Because gradients are orthogonal to level curves, this means that grad H is tangent to C at (x, y) Consequently, if N denotes a unit vector normal to C at (x,y), grad H is orthogonal to N. That is,

(grad H) ■ N=0.

Finally, since

dH = (grad H) ■ N, dN B 2

we may conclude from equation that dH/dN=0 at points on C.

In this discussion, we have tacitly assumed that grad h=0. If grad h=0, it follows from the identity

Igrad H(x,y)\=Igrad h(u,v)\\f '(z)\,

derived in of this section, that grad H=0; hence dh/dn and the corresponding normal derivative dH/dN are both zero. We have also assumed that

grad h and grad H always exist;

the level curve H(x, y)=c is smooth when grad h=0 at (u, v).

Condition ensures that angles between arcs are preserved by transformation when it is conformal. In all of our applications be satisfied.

EXAMPLE. Consider, for instance, the function h(u, v)=v + 2. The transformation

w=iz2=—2xy + i(x2 — y2)

is conformal when z=0. It maps the half line y=x (x > 0) onto the negative u axis, where h=2, and the positive x axis onto the positive v axis, where the normal derivative hu is 0. According to the above theorem, the function

$H(x,y)=x^2 - y^2 + 2$

must satisfy the condition H=2 along the half line y=x (x > 0) and Hy=0 along the positive x axis, as one can verify directly.

A boundary condition that is not of one of the two types mentioned in the theorem may be transformed into a condition that is substantially different from the original one New boundary conditions for the transformed problem can be obtained for a particular transformation in any case. It is interesting to note that under a conformal transformation, the ratio of a directional derivative of H along a smooth arc C in the z plane to the directional derivative of h along the image curve Y at the corresponding point in the w plane is \f'(z)\; usually, this ratio is not constant along a given arc.

EXERCISES

Use expression to find a harmonic conjugate of the harmonic function $u(x,y)=x^3 - 3xy^2$. Write the resulting analytic function in terms of the complex variable z.

Let u(x, y) be harmonic in a simply connected domain D. By appealing to results show that its partial derivatives of all orders are continuous throughout that domain.

The transformation w=exp z maps the horizontal strip 0 < y < n onto the upper half plane v > 0, and the function

$h(u, v)=Re(w^2)=u^2 - v^2$

is harmonic in that half plane. show that the function $H(x, y)=e^{2x} \cos 2y$ is harmonic in the strip. Verify this result directly.

Under the transformation w=exp z, the image of the segment 0 < y < n of the y axis is the semicircle $u^2 + v^2=1, v > 0$ (see Sec. 14). Also, the function

h{u, v)=Re(^2 — w H^=2 — u H— y

Notes

\ w/ u2 + v2

is harmonic everywhere in the w plane except for the origin; and it assumes the value h=2 on the semicircle. Write an explicit expression for the function H(x, y) Then illustrate the theorem by showing directly that H=2 along the segment 0 < y < n of the y axis.

The transformation w=$_{Z2}$ maps the positive x and y axes and the origin in the z plane onto the u axis in the w plane. Consider the harmonic function

h(u, v)=Re(e—w)=e— cos v,

and observe that its normal derivative hv along the u axis is zero. Then illustrate the when f(z)=$_{Z2}$ by showing directly that the normal derivative of the function H(x, y) defined in that theorem is zero along both positive axes in the z plane. (Note that the transformation w=$_{Z2}$ is not conformal at the origin.)

Replace the function h(u, v) in by the harmonic function

h(u, v)=Re(-2iw + e—w)=2v + e—u cos v.

Then show that hv=2 along the u axis but that Hy=4x along the positive x axis and Hx=4y along the positive y axis. This illustrates how a condition of the type is not necessarily transformed into a condition of the type dH/dN=ho.

Show that if a function H(x, y) is a solution of a Neumann H(x,y) + A, where A is any real constant, is also a solution of that problem.

Suppose that an analytic function w=f(z)=u(x,y) + iv(x,y) maps a domain Dz in the z plane onto a domain Dw in the w plane; and let a function h(u, v), with continuous partial derivatives of the first and second order, be defined on Dw. Use the chain rule for partial derivatives to show that if H(x, y)=h[u(x, y), v(x, y)], then

Hxx(x, y) + Hyy(x, y)=[huu(u, v) + hvv(u, v)] |f'(z)|2.

Conclude that the function $H(x,y)$ is harmonic in D_z when $h(u, v)$ is harmonic in D_w. This is an alternative proof of the theorem in Sec. 105, even when the domain D_w is multiply connected.

Suggestion: In the simplifications, it is important to note that since f is analytic, the Cauchy-Riemann equations $u_x=v_y, u_y=—v_x$ hold and that the functions u and v both satisfy Laplace's equation. Also, the continuity conditions on the derivatives of h ensure that $h_{vu}=h_{uv}$.

Let $p(u,v)$ be a function that has continuous partial derivatives of the first and second order and satisfies Poisson's equation

$p_{uu}(u, v) + p_{vv}(u, v)=<\&(u, v)$

in a domain D_w of the w plane, where ® is a prescribed function. Show how it follows from the identity obtained in Exercise 8 that if an analytic function

$w=f(z)=u(x, y) + iv(x, y)$

maps a domain D_z onto the domain D_w, then the function

$P(x, y)=p[u(x, y), v(x, y)]$

satisfies the Poisson equation

$P_{xx}(x, y) + P_{yy}(x, y)=$®$[u(x, y), v(x, y)]$ \f '{z)\2 in D_z.

Suppose that $w=f(z)=u(x, y) + iv(x, y)$ is a conformal mapping of a smooth arc C onto a smooth arc T in the w plane. Let the function $h(u, v)$ be defined on T, and write

$H(x, y)=h[u(x, y), v(x, y)]$.

From calculus, we know that the x and y components of grad H are the partial derivatives H_x and H_y, respectively; likewise, grad h has components h_u and h_v. By applying the chain rule for partial derivatives and using the Cauchy-Riemann equations, show that if (x, y) is a point on C and (u, v) is its image on T, then \grad H(x, y)\=|grad h(u, v)\\f"(z)\.

13.9 TWO-DIMENSIONAL FLUID FLOW

Notes

Harmonic functions play an important role in hydrodynamics and aerodynamics. Again, we consider only the two-dimensional steady-state type of problem. That is, the motion of the fluid is assumed to be the same in all planes parallel to the xy plane, the velocity being parallel to that plane and independent of time. It is, then, sufficient to consider the motion of a sheet of fluid in the xy plane.

We let the vector representing the complex number

V=p + iq denote the velocity of a particle of the fluid at any point (x,y); hence the x and y components of the velocity vector are p(x,y) and q(x,y), respectively. At points interior to a region of flow in which no sources or sinks of the fluid occur, the real-valued functions p(x,y) and q(x,y) and their first-order partial derivatives are assumed to be continuous.

The circulation of the fluid along any contour C is defined as the line integral with respect to arc length a of the tangential component VT(x,y) of the velocity vector along C:

J Vt(x, y) da.

The ratio of the circulation along C to the length of C is, therefore, a mean speed of the fluid along that contour. It is shown in advanced calculus that such an integral can be written

/ Vt(x, y) da=p(x,y) dx + q(x,y) dy.

When C is a positively oriented simple closed contour lying in a simply connected domain of flow containing no sources or sinks, Green's theorem enables us to write

/ p(x,y) dx + q(x,y) dy=[qx(x,y) - py(x,y)] dA, Jc J Jr

where R is the closed region consisting of points interior to and on C. Thus

/ Vt(x, y) da=/ [qx(x,y) - py(x,y)] dA Jc J Jr

for such a contour

A physical interpretation of the integrand on the right in expression for the circulation along the simple closed contour C is readily given. We let C denote a circle of radius r which is centered at a point (x0, y0) and taken counterclockwise.

The mean speed along C is then found by dividing the circulation by the circumference 2nr, and the corresponding mean angular speed of the fluid about the center of the circle is obtained by dividing that mean speed by r:

1 IIr \^lx(x'y) ~Py(x-y^dA- nr2 J Jr 2

Now this is also an expression for the mean value of the function

co(x, y)=^ [qx(x, y) - py(x, y)]

over the circular region R bounded by C. Its limit as r tends to zero is the value of rn at the point (x0, y0). Hence the function rn(x, y), called the rotation of the fluid, represents the limiting angular speed of a circular element of the fluid as the circle shrinks to its center (x,y), the point at which rn is evaluated.

If rn(x,y)=0 at each point in some simply connected domain, the flow is irrotational in that domain. We consider only irrotational flows here, and we also assume that the fluid is incompressible and free from viscosity. Under our assumption of steady irrotational flow of fluids with uniform density p, it can be shown that the fluid pressure P(x,y) satisfies the following special case of Bernoulli's equation:

- + \\y\2=c p2

where c is a constant. Note that the pressure is greatest where the speed | V | is least.

Let D be a simply connected domain in which the flow is irrotational. Accord- ing to equation py=qx throughout D. This relation between partial derivatives implies that the line integral (xo, yo) is changed. The level curves $(x, y)=c\ are called equipotentials. Because it is the gradient of $(x,y), the velocity vector V is normal to an equipotential at any point where V is not the zero vector.

Just as in the case of the flow of heat, the condition that the incompressible fluid enter or leave an element of volume only by flowing through the boundary of that element requires that $\$(x,y)$ must satisfy Laplace's equation

$$<Pxx(x, y) + \&yy(x, y)=0$$

in a domain where the fluid is free from sources or sinks. In view of equations and the continuity of the functions p and q and their first-order partial derivatives, it follows that the partial derivatives of the first and second order of $\$$ are continuous in such a domain. Hence the velocity potential $\$$ is a harmonic function in that domain.

13.10 THE STREAM FUNCTION

According to the velocity vector

$$V=p(x, y) + iq(x, y)$$

for a simply connected domain in which the flow is irrotational can be written

$$V=\$x(x, y) + i\$y(x, y)=grad\ \$(x, y),$$

where $\$$ is the velocity potential. When the velocity vector is not the zero vector, it is normal to an equipotential passing through the point (x,y). If, moreover, f(x,y) denotes a harmonic conjugate of $\$(x, y)$, the velocity vector is tangent to a curve f(x,y)=C2. The curves f(x,y)=C2 are called the streamlines of the flow, and the function f is the stream function. In particular, a boundary across which fluid cannot flow is a streamline.

The analytic function

$F(z)=\$(x,y)+if(x,y)$ is called the complex potential of the flow. Note that

$F'(z)=\$x(x,y)+ifx(x,y)$ and, in view of the Cauchy-Riemann equations,

$F'(z)=\$x(x,y)-i\$y(x,y)$. Expression for the velocity thus becomes

$$V=F\ '(z).$$

The speed, or magnitude of the velocity, is obtained by writing

|V |=\F'(z)\.

According to equation if <p is harmonic in a simply connected domain D, a harmonic conjugate of $ there can be written

r (x,y)

ty(x,y)=I -$t(s,t)ds + $s(s,t) dt,

J (x0,y0)

where the integration is independent of path. With the aid of equations we can, therefore, write

f(x,y)=L -q(s, t) ds + p(s, t) dt,

where C is any contour in D from (x0, y0) to (x,y).

Now it is shown in advanced calculus that the right-hand side of equation represents the integral with respect to arc length a along C of the normal compo- nent VN(x, y) of the vector whose x and y components are p(x,y) and q(x,y), respectively. So expression can be written

f(x,y)=C Vn(s, t) da.

Physically, then, ty(x,y) represents the time rate of flow of the fluid across C. More precisely, ty(x,y) denotes the rate of flow, by volume, across a surface of unit height standing perpendicular to the xy plane on the curve C.

EXAMPLE. When the complex potential is the function

F(z)=Az , where A is a positive real constant,

$(x, y)=Ax and ^(x, y)=Ay.

The streamlines ty(x,y)=C2 are the horizontal lines y=C2/A, and the velocity at any point is

V =1^U)=A.

Here a point (x0, y0) at which ft(x,y)=0 is any point on the x axis. If the point (x0, y0) is taken as the origin, then ft(x,y) is the rate of flow across any contour drawn from the origin to the point (x, y). The flow is uniform and to the right. It can be interpreted as the uniform flow in the upper half plane bounded by the x axis, which is a streamline, or as the uniform flow between two parallel lines y=y1 and y=y^y

(yy) T V

The stream function ty characterizes a definite flow in a region. The question of whether just one such function exists corresponding to a given region, except possi- bly for a constant factor or an additive constant, is not examined here. Sometimes, when the velocity is uniform far from the obstruction or when sources and sinks are involved, the physical situation indicates that the flow is uniquely determined by the conditions given in the problem.

A harmonic function is not always uniquely determined, even up to a constant factor, by simply prescribing its values on the boundary of a region. In the example above, the function ty(x,y)=Ay is harmonic in the half plane y > 0 and has zero values on the boundary. The function ty\(x, y)=Be8siny also satisfies those conditions. However, the streamline ty\(x,y)=0 consists not only of the line y=0 but also of the lines y=nn (n=1, 2,...). Here the function F\(z)=Bez is the complex potential for the flow in the strip between the lines y=0 and y=n, both lines making up the streamline ty(x,y)=0; if B > 0, the fluid flows to the right along the lower line and to the left along the upper one.

13.11 FLOWS AROUND A CORNER AND AROUND A CYLINDER

In analyZ1ng a flow in the xy, or z, plane, it is often simpler to consider a corresponding flow in the uv, or w, plane. Then, if is a velocity potential and ty a stream function for the flow in the uv plane, can be applied to these harmonic functions. That is, when the domain of flow Dw in the uv plane is the image of a domain Dz under a transformation

w=f(z)=u(x, y) + iv(x, y), where f is analytic, the functions

$[u(x,y),v(x,y)] and ty [u(x, y), v(x, y)]

are harmonic in Dz. These new functions may be interpreted as velocity potential and stream function in the xy plane. A streamline or natural boundary ty(u,v)=ci in the uv plane corresponds to a streamline or natural boundary ty[u(x, y), v(x, y)]=C2 in the xy plane.

In using this technique, it is often most efficient to first write the complex potential function for the region in the w plane and then obtain from that the velocity potential and stream function for the corresponding region in the xy plane. More precisely, if the potential function in the uv plane is

F(w)=$(u, v) + ity(u, v),

the composite function

F[f(z)]=$[u(x, y), v(x, y)] + ity[u(x, y), v(x, y)] is the desired complex potential in the xy plane.

In order to avoid an excess of notation, we use the same symbols F, <p, and ^for the complex potential, etc., in both the xy and the uv planes.

EXAMPLE . Consider a flow in the first quadrant x > 0, y > 0 that comes in downward parallel to the y axis but is forced to turn a corner near the origin,. To determine the flow, that the transformation

2 2 2 , •<-> w=z=x — y + i2xy

maps the first quadrant onto the upper half of the uv plane and the boundary of the quadrant onto the entire u axis. From complex potential for a uniform flow to the right in the upper half of the w plane is F=Aw, where A is a positive real constant. The potential in the quadrant is, therefore,

F=Az2=A(x2 — y2) + i2Axy; and it follows that the stream function for the flow there is

Notes

$= 2Axy$.

This stream function is, of course, harmonic in the first quadrant, and it vanishes on the boundary.

The streamlines are branches of the rectangular hyperbolas

2Axy=C2.

According to equation, the velocity of the fluid is

V=2 Az=2A(x — iy).

Observe that the speed

|V|=2 A^Jx1 + v2

of a particle is directly proportional to its distance from the origin. The value of the stream function at a point (x, y) can be interpreted as the rate of flow across a line segment extending from the origin to that point.

EXAMPLE. Let a long circular cylinder of unit radius be placed in a large body of fluid flowing with a uniform velocity, the axis of the cylinder being perpendicular to the direction of flow. To determine the steady flow around the cylinder, we represent the cylinder by the circle x2 + y2=1 and let the flow distant from it be parallel to the x axis and to the right. Symmetry shows that points on the x axis exterior to the circle may be treated as boundary points, and so we need to consider only the upper part of the figure as the region of flow.

The boundary of this region of flow, consisting of the upper semicircle and the parts of the x axis exterior to the circle, is mapped onto the entire u axis by the transformation

The region itself is mapped onto the upper half plane v > 0, as indicated The complex potential for the corresponding uniform flow in that half plane is F=Aw, where A is a positive real constant. Hence the complex potential for the region exterior to the circle and above the x axis is The velocity approaches A as |z| increases. Thus the flow is nearly uniform and parallel to the x axis at points distant from the circle, as one would expect. From expression we see that V(z)=V(z): hence that expression also represents velocities of flow in the lower region, the lower semicircle being a streamline.

According to equation, the stream function for the given problem is, in polar coordinates,

1fr=A[r) sin0.

The streamlines

A (r) sin 6>=c2

are symmetric to the y axis and have asymptotes parallel to the x axis. Note that when C2=0, the streamline consists of the circle r=1 and the parts of the x axis exterior to the circle.

Check your Progress-1

Discuss Conformal Mapping

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Discuss Transformations of Boundary Conditions

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13.12 LET US SUM UP

In this unit we have discussed the definition and example of Conformal Mapping, Preservation Of Angles, Scale Factors, Local Inverses, Harmonic Conjugates, Transformations Of Harmonic Functions, Transformations Of Boundary Conditions, Two-Dimensional Fluid Flow, The Stream Function, Flows Around A Corner And Around A Cylinder

13.13 KEYWORDS

Conformal Mapping In this chapter, we introduce and develop the concept of a conformal mapping, with emphasis on connections between such mappings and harmonic functions

Preservation Of Angles Let C be a smooth arc represented by the equation z=z(t) (a < t < b)

Scale Factors Another property of a transformation w=f(z) that is conformal at a point z0 is obtained by considering the modulus of f '(zo)

Local Inverses A transformation w=f(z) that is conformal at a point Z0 has a local inverse there. That is, if W0=f(z0), then there exists a unique transformation z=g(w)

Harmonic Conjugates If a function f(z)=u(x, y) + iv(x, y) is analytic in a domain D

Transformations Of Harmonic Functions The problem of finding a function that is harmonic in a specified domain and satisfies prescribed conditions on the boundary of the domain is prominent in applied mathematics

Transformations Of Boundary Conditions The conditions that a function or its normal derivative have prescribed values along the boundary of a domain in which it is harmonic are the most common, although not the only, important types of boundary conditions.

Two-Dimensional Fluid Flow Harmonic functions play an important role in hydrodynamics and aerodynamics.

The Stream Function According to the velocity vector V=p(x, y) + iq(x, y)

Flows Around A Corner And Around A Cylinder In analyZ1ng a flow in the xy, or z, plane, it is often simpler to consider a corresponding flow in the uv, or w, plane.

13.14 QUESTIONS FOR REVIEW

Explain Conformal Mapping

Explain Transformations Of Boundary Conditions

13.15 ANSWERS TO CHECK YOUR PROGRESS

Conformal Mapping (answer for Check your Progress-1 Q)

Transformations of Boundary Conditions

(answer for Check your Progress-1 Q)

13.16 REFERENCES

- Basic of Complex Analysis
- Introduction To Complex Analysis
- Application Of Complex Analysis & Variables
- Complex Functions & Variables
- Complex Variables,Complex Functions

UNIT-14 : SCHWARZ-CHRISTOFFEL TRANSFORMATION

STRUCTURE

14.0 OBJECTIVES

After studying this unit, you should be able to:

Learn, Understand about Schwarz-Christoffel Transformation

Triangles And Rectangles

Integral Formulas Of The Poisson Type

Poisson Integral Formula

Dirichlet Problem For A Disk

Schwarz Integral Formula

Dirichlet Problem For A Half Plane

Neumann Problems

14.1 INTRODUCTION

In this part of the course we will study some basic complex analysis . This is an extremely useful and beautiful part of mathematics and forms the basis of many techniques employed in many branches of mathematic In this section we will study complex functions of a complex variable, Schwarz-Christoffel Transformation, Triangles And Rectangles, Integral Formulas Of The Poisson Type, Poisson Integral Formula, Dirichlet Problem For A Disk, Schwarz Integral Formula, Dirichlet Problem For A Half Plane, Neumann Problems

14.2 SCHWARZ-CHRISTOFFEL TRANSFORMATION

In our expression f'(z)=A(z — X1)—k1 (z — X2)—k2 ■■■(z — Xn — 1)—kn—1

for the derivative of a function that is to map the x axis onto a polygon, let the factors (z — Xj)—kj (j=1, 2,... ,n — 1) represent branches of power functions with branch cuts extending below that axis. To be specific, write

(z — Xj)—kj=exp[—kj log(z — Xj)]=exp[—kj(ln |z — Xj | + iQj)]

and then

(z - Xj)~ki=|z - Xj\~ki expl-ikjOj)   < 9j < ,

where Qj=arg(z — Xj) and j=1, 2,... ,n — 1. This makes f'(z) analytic every where in the half plane y > 0 except at the n — 1 branch points Xj.

Notes

If z0 is a point in that region of analyticity, denoted here by R, then the function

F(z)=f f '(s) ds J10

is single-valued and analytic throughout the same region, where the path of inte- gration from z0 to z is any contour lying within R. Moreover, F'(z)=f '(z) To define the function F at the point z=X1 so that it is continuous there, we note that (z — X1)—k1 is the only factor in expression that is not analytic at X1. Hence if $(z) denotes the product of the rest of the factors in that expression, $(z) is analytic at the point X1 and is represented throughout an open disk $|Z - X1|$ < R1 by its Taylor series about X1. So we can write

4>'(Xl) $ "(X\)2

0(-vr) H — U-jtiH y~(z~Xl) + or

f '(z)=<p(xi)(z — xi)"kl + (z — xi)1-kl f(z),

where ^ is analytic and therefore continuous throughout the entire open disk. Since l — k1 > 0, the last term on the right in equation thus represents a continuous function of z throughout the upper half of the disk, where Im z > 0, if we assign it the value zero at z=X1. It follows that the integral

f (s — x1)1—k1 f(s) ds Jz1

of that last term along a contour from z_1 to z , where z_1 and the contour lie in the half disk, is a continuous function of z at z=X1. The integral

(*-*!)-*! ds= -Uz - -VI)!-A'l - (z_1 - JTi)1-*!]

JZX 1 — k1

along the same path also represents a continuous function of z at X1 if we define the value of the integral there as its limit as z approaches x1 in the half disk. The integral of the function along the stated path from z_1 to z is, then, continuous at z=X1; and the same is true of integral since it can be written as an integral along a contour in R from z0 to z_1 plus the integral from z_1 to z.

The above argument applies at each of the n — 1 points Xj to make F contin- uous throughout the region y > 0.

From equation we can show that for a sufficiently large positive number R, a positive constant M exists such that if Im z > 0, then

M

|/ (z)| < kn whenever |z| > R.

Since 2 — kn > 1, this order property of the integrand in equation ensures the existence of the limit of the integral there as z tends to infinity; that is, a number Wn exists such that

lim F(z)=Wn (Im z > 0).

Our mapping function, whose derivative is given by equation, can be written f(z)=F(z) + B, where B is a complex constant. The resulting transformation,

w=a((s — X1)~k1 (s — X2)~kz ■■■ (s — Xn—1)—kn—1 ds + B,

is the Schwarz-Christoffel transformation, named in honor of the two German math ematicians H. A. Schwarz (1843-1921) and E. B. Christoffel (1829-1900) who discovered it independently.

Transformation is continuous throughout the half plane y > 0 and is conformal there except for the points Xj. We have assumed that the numbers kj satisfy

conditions. In addition, we suppose that the constants xj and kj are such that the sides of the polygon do not cross, so that the polygon is a simple closed contour. Then, according to, as the point z describes the x axis in the positive direction, its image w describes the polygon P in the positive sense; and there is a one to one correspondence between points on that axis and points on P. According to condition, the image wn of the point z=*x> exists and wn=Wn + B.

If z is an interior point of the upper half plane y > 0 and X0 is any point on the x axis other than one of the xj, then the angle from the vector t at X0 up to the line segment joining x0 and z is positive and less than n. At the image W0 of x0, the corresponding angle from the vector t to the image of the line segment joining x_0 and z has that same value. Thus the images of interior points in the half plane lie to the left of the sides of the polygon, taken counterclockwise. A proof that the transformation establishes a one to one correspondence between the interior points of the half plane and the points within the polygon is left to the reader

Given a specific polygon P, let us examine the number of constants in the Schwarz-Christoffel transformation that must be determined in order to map the x axis onto P. For this purpose, we may write $z_0=0$, A=1, and B=0 and simply require that the x axis be mapped onto some polygon P' similar to P. The size and position of P' can then be adjusted to match those of P by introducing the appropriate constants A and B.

The numbers k_j are all determined from the exterior angles at the vertices of P . The n — 1 constants x_j remain to be chosen. The image of the x axis is some polygon P' that has the same angles as P. But if P' is to be similar to P, then n — 2 connected sides must have a common ratio to the corresponding sides of P; this condition is expressed by means of n — 3 equations in the n — 1 real unknowns x_j. Thus two of the numbers x_j, or two relations between them, can be chosen arbitrarily, provided those n — 3 equations in the remaining n — 3 unknowns have real-valued solutions.

When a finite point $z=x_n$ on the x axis, instead of the point at infinity, represents the point whose image is the vertex w_n, it follows from tthat the Schwarz-Christoffel transformation takes the form

where $k_1 + k_2 + \ldots + k_n=2$. The exponents k_j are determined from the exterior angles of the polygon. But, in this case, there are n real constants x_j that must satisfy the n — 3 equations noted above. Thus three of the numbers x_j, or three conditions on those n numbers, can be chosen arbitrarily when transformation is used to map the x axis onto a given polygon.

14.3 TRIANGLES AND RECTANGLES

The Schwarz-Christoffel transformation is written in terms of the points x_j and not in terms of their images, which are the vertices of the polygon.

No more than three of those points can be chosen arbitrarily; so, when the given polygon has more than three sides, some of the points xj must be determined in order to make the given polygon, or any polygon similar to it, be the image of the x axis. The selection of conditions for the determination of those constants that are convenient to use often requires ingenuity.

Another limitation in using the transformation is due to the integration that is involved. Often the integral cannot be evaluated in terms of a finite number of elementary functions. In such cases, the solution of problems by means of the transformation can become quite involved.

If the polygon is a triangle with vertices at the points W1, W2, and W3 the transformation can be written

w=A ((s — xi)—k1 (s — X2)~k2 (s — X3)~k3 ds + B,

where ki + Ay + £3=2. In terms of the interior angles Qj,

kj=l--9j (7=1,2,3). n

Here we have taken all three points xj as finite points on the x axis. Arbitrary values can be assigned to each of them. The complex constants A and B, which are associated with the size and position of the triangle, can be determined so that the upper half plane is mapped onto the given triangular region.

If we take the vertex W3 as the image of the point at infinity, the transformation becomes

w=a[(s — x1)—k1 (s — x2)—k2 ds + B,

Jzo

where arbitrary real values can be assigned to x1 and x2.

The integrals in equations do not represent elementary functions unless the triangle is degenerate with one or two of its vertices at infinity. The integral in equation becomes an elliptic integral when the triangle is equi-lateral or when it is a right triangle with one of its angles equal to either n/3 or n/4.

Notes

EXAMPLE. For an equilateral triangle, k\=kz=k=2/3. It is conve- nient to write x\=— 1, X2=1, and X3=<x and to use equation, with Z0=1, A=1, and B=0. The transformation then becomes

w=J\s + 1)—2/3(s — 1)—2/3 ds.

The image of the point z=1 is clearly w=0; that is, W2=0. If z=— 1 in this integral, one can write s=x, where —1 < x < 1. Then

x + 1 > 0 and arg(x + 1)=0,

while

|x — 1|=1 — x and arg(x — 1)=n.

Hence

w= J (x + 1) —2/3(i _ ,v)~2/3 exp^— dx

(ni\C1 2 dx = eXPUJio

when z=—l- With the substitution a=+/t, the last integral here reduces to a special case of the one used in defining the beta function. Let b denote its value, which is positive:

The vertex W1 is, therefore, the point

uq=b exp —.

The vertex W3 is on the positive u axis because

U'3=/ (v + 1)—2/3(.v — l)-2/3 dx={x2d_\)Z/3-

But the value of W3 is also represented by integral when z tends to infinity along the negative x axis; that is,

j^ (\x + 1||-.v - 1|) 2/3exp^--^-^v

+ J ^ (k + 111-v — 1|)_2/3 exp^—dx.

In view of the first of expressions for W1, then,

u>3=wi + exp^-^-^ j (|jt + l|k - 1|)~2/3dx ni (ni\ dx

= i,ex fY+exe{-T)lirvipvs-or

ni / ni\

w3=b exp — + w3 expl —— I.

Solving for W3, we find that

W3=b.

We have thus verified that the image of the x axis is the equilateral triangle of side We can also see that

b ni w=— exp — when ~=0.

When the polygon is a rectangle, each kj=1/2. If we choose ±1 and ±a as the points xj whose images are the vertices and write

g(z)=(z + a)—1/2(z + 1)-1/2(z - 1)-1/2(z - a)-1/2,

where 0 < arg(z — xj) < n, the Schwarz-Christoffel transformation becomes

w=— f g(s) ds,

Jo

except for a transformation W=Aw + B to adjust the size and position of the rectangle. Integral is a constant times the elliptic integral

j\l-s2)-l/2(l-k2s2)-l/2 ds (*=£)

but the form of the integrand indicates more clearly the appropriate branches of the power functions involved.

EXAMPLE . Let us locate the vertices of the rectangle when a > 1, x\=—a, X2=—1, X3=1, and X4=a. All four vertices can be described in terms of two positive numbers b and c that depend on the value of a in the following manner :

dx

b=|g(x)| dx =

(|g(x)| dx=f —

Jo Jo J(1

fa fa dx

C=Ji dx=J

y (x2 — 1)(a2 — x2)

If — 1 < x < 0, then

arg(x + a)=arg(x + 1)=0 and arg(x — 1)=arg(x — a)=n; hence

/ 71A1 |y(.r)|=|y(x)|.

g(x) = If —a < x < —1, then ni expl -y

I g(x) |=i\g(x)\.

Thus

|g(x) | dx — i I |g(:x) | dx=—b + ic.

It is left to the exercises to show that W2=—b, W3=b, W4=b + ic.

The position and dimensions of the rectangle

14.4 INTEGRAL FORMULAS OF THE POISSON TYPE

In this chapter, we develop a theory that enables us to solve a variety of boundary value problems whose solutions are expressed in terms of definite or improper integrals. Many of the integrals occurring are then readily evaluated.

14.5 POISSON INTEGRAL FORMULA

Let C0 denote a positively oriented circle, centered at the origin, and suppose that a function f is analytic inside and on C0. The Cauchy integral formula expresses the value of f at any point z interior to C0 in terms of the values of f at points s on Q. In this section, we shall obtain from formula a corresponding formula for the real component of the function f; and,we shall use that result to solve the Dirichlet problem for the disk bounded by C0.

We let r0 denote the radius of C0 and write z=r exp(76>), where 0 < r < r$ The inverse of the nonzero point z with respect to the circle is the point z_1 lying on the same ray from the origin as z and satisfying the condition |z_1 ||z|=r^. Because (r0/r) > 1,

and this means that z_1 is exterior to the circle Co. According to the Cauchy-Goursat theorem, then,

f(s)ds L

/Co s — z_1 Hence

f(z)=±t (— —)f(s)ds-,

2ni Jc0\s - z s - Z1j

and, using the parametric representation s=roexp(i^) (0 < <p < 2n) for Co, we have

f (z)=—— f —\f(s)d(p

2n Jo \s - z s - Z1j

where, for convenience, we retain the s to denote ro exp(ij>). Now

4 je _ 4 _ z_1=—e =r

and, in view of this expression for z_1 , the quantity inside the parentheses in equation

can be written

— 2 2 i^=-?- + J^=r*-r

s—z s—s(s/z) s—z ' s—z $\frac{|s-z|}{z}$ An alternative form of the Cauchy integral formula is, therefore,

f ^ ,, 2n Jo |s - z|2

when 0 < r < ro. This form is also valid when r=0; in that case, it reduces directly to

1 C 2^ f{0)=^T 2n Jo

Notes

which is just the parametric form of equation with z=0. The quantity \s — z\ is the distance between the points s and z, and the law of cosines can be used to write

\s — z\2=r0 — 2r0r cos(p — 6) + r2.

Hence, if u is the real component of the analytic function f, it follows from formula that

1 f2n (r02 — r 2)u(r0,4>)

n(r,0)=— I dtp (r < ro).

2^ Jo rfi — 2ror cos(tp — 9) + r2

This is the Poisson integral formula for the harmonic function u in the open disk bounded by the circle r=r0.

Formula defines a linear integral transformation of u(r0 ,p) into u(r,6). The kernel of the transformation is, except for the factor 1/(2n), the real-valued function

P(ro, r, (p - 9) =

r0 — 2r0r cos(f — 6) + r2 ' which is known as the Poisson kernel. In view of equation, we can also write

r2 r2

P(ro,r,<p-9)=0 •s — z\2'

and, since r < ro, it is clear that P is a positive function. Moreover, since z/Cs —z) and its complex conjugate z/(s — z) have the same real parts, we find from the second of equations that

P(r0, r,<p — 9)=Re(^— 1 —^=Re^ ^^

s — z s — z s — z

Thus P(r0,r,$ — 6) is a harmonic function of r and 6 interior to C0 for each fixed s on C0. From equation, we see that P(r ,r,p> — 6) is an even periodic function of <p — 6, with period 2n, and that its value is 1 when r=0.

The Poisson integral formula can now be written

14.6 DIRICHLET PROBLEM FOR A DISK

Let F be a piecewise continuous function of Q on the interval 0 < Q < 2n. The Poisson integral transform of F is defined in terms of the Poisson kernel P(r0, r , p — Q), introduced by means of the equation

1 f2n lJ(r,6)=-— I P(r0.r.p - 0)F(p) dp (r < r0).

2n J0

In this section, we shall prove that the function U(r , Q) is harmonic inside the circle r=r0 and

lim U(r, Q)=F(Q)

r^r0 r<r0

for each fixed Q at which F is continuous. Thus U is a solution of the Dirichlet problem for the disk r < r0 in the sense that U(r,Q) approaches the boundary value F(Q) as the point (r, Q) approaches (r0,Q) along a radius, except at the finite number of points (r0,Q) where discontinuities of F may occur.

EXAMPLE. Before proving the statement in italics, let us apply it to find the potential V(r,Q) inside a long hollow circular cylinder of unit radius, split lengthwise into two equal parts, when V=1 on one of the parts and V=0 on the other. This problem was solved by conformal mapping; and we recall how it was interpreted there as a Dirichlet problem for the disk r < 1, where V=0 on the upper half of the boundary r=1 and V=1 on the lower half. y

V=0

V=1

In equation, write V for U, r0=1, and

}0 when 0 < < n, ()=11 when n < (p < 2n

to obtain

-2n 1 r2

V(r,9)=—J P(l,r,<p-e)d<p, where

1 — r2 P(hr,4>-9)= ——.

1 + r2 — 2r cos(<p — 9)

An antiderivative of P(1,r,ty) is

j P(1, r, \fr)d\fr=2 aiclan^ | ^ ' tan ,

the integrand here being the derivative with respect to ^ of the function on the right. So it follows from expression that

(1 + r 2n — 9 \ (1 + r n — 9

nV(r,9)=arctan tan I — arctan tan

\1 — r 2 j \1 — r 2

After simplifying the expression for tan[nV(r, 9)] obtained from this last equation, we find that

1 (1 r2 \

V(r,9)=- arctan) (0 < arctanf < n),

n \2r sin9 j

where the stated restriction on the values of the arctangent function is physically evident. When expressed in rectangular coordinates, the solution here is the same as solution.

We turn now to the proof that the function U defined in equation satisfies the Dirichlet problem for the disk r < r^, as asserted just prior to this example. First of all, U is harmonic inside the circle r=r$ because P is a harmonic function of r and 9 there. More precisely, since F is piecewise continuous, integral can be written as the sum of a finite number of definite integrals each of which has an integrand that is continuous in r, 9, and The partial derivatives of those integrands with respect to r and 9 are also continuous. Since the order of integration and differentiation with respect to r and 9 can, then, be interchanged and since P satisfies Laplace's equation

r2 Prr + rPr + P99=0

in the polar coordinates r and 9, it follows that U satisfies that equation too.

In order to verify limit, we need to show that if F is continuous at 9, there corresponds to each positive number e a positive number S such that

\U(r, 9) — F(9)\ < e whenever 0 < r0 — r < S.

We start by referring to property of the Poisson kernel and writing

1 r2n U(r,6) — F(6)=— I P(r,,, r,<p~9) [F(4» - F(9)] dep.

2n J0

For convenience, we let F be extended periodically, with period 2n, so that the integrand here is periodic in $ with that same period. Also, we may assume that 0 < r < ro because of the nature of the limit to be established.

Next, we observe that since F is continuous at Q, there is a small positive number a such that

|F(ep) — F(9)| < - whenever \ep — 9\ < a.

Evidently,

U(r,Q) - F(Q)=h(r) + I2(r) where

1 i-Q+a

h(r)=— P(r0, r9<f> — 9) [F(</>) — F(#)] d(p,

2n Jq-a 1 r Q -a+2n

h(r)=— P(r0, r.ep — 9) [F(<P) - F(9)] dep.2n JQ+a

The fact that P is a positive function, together with the first of inequalities of that function, enables us to write

1 r Q+a \h(r)\ < — P(r0,r,4>-9)\F(4»-F(9)\d4>

Notes

2n Jq-a £ f2n £

^ to J0 P{ro,r-^ ~6) d<^=2'

As for the integral h(r), that the denominator |s - z|2 in expression for P(ro,r,$ - Q) in that section has a (positive) minimum value m as the argument $ of s varies over the closed interval

Q + a < $ < Q - a + 2n.

So, if M denotes an upper bound of the piecewise continuous function \F($) - F(Q)\ on the interval 0 < $ < 2n, it follows that

(rQ2 - r2)M 2Mro 2Mro £

\h(r)\ < 2ti < (r0 - r) < S=-

2nm m m 2

whenever ro - r < 8 where

me

S =

Mro

Finally, the results in the two preceding paragraphs tell us that

ee

\U(r,9)-F(9)\ < |/i(r)| + |/2(r)| < - + - =e

whenever r0 — r < S, where S is the positive number defined by equation That is, statement holds when that choice of S is made.

According to expression and since P(r0, 0,p — 9)=1,

U(0,9)=^~ [2n J0

Thus the value of a harmonic function at the center of the circle r=r0 is the average of the boundary values on the circle.

It is left to the exercises to prove that P and U can be represented by series involving the elementary harmonic functions rn cos n9 and rn sin n9 as follows

^/r V

P(ro,r,<p - Q)=l + 2^(—) cos n(p-Q) (r < r0)

n=1 ^ 0' ^ (r \n

U(r,9)=-ao + (— I (an cosn9 + b,, sinnt?) (r < ro),

i\r0/ n=1 N 7

where 1 f2n 1 f2n

an=— I F (<p) cos tup dip, bn=— I F (ip) sintup dip.

n J0 n J0

for the electrostatic potential interior to a cylinder x2 + y2=1 when V=1 on the first quadrant (x > 0,y > 0) of the cylindrical surface and V=0 on the rest of that surface. Also, point out why 1 — V is the solution

Let T denote the steady temperatures in a disk r < 1, with insulated faces, when T=1 on the arc 0 < 9 < 290 (0 <90 < n/2) of the edge r=1 and T=0 on the rest of the edge. Use the Poisson integral transform to show that

(1 _ x2 _ y2)y(j

(x - l)2 + (V - yo)2 - }'o

where y0=tan 90 . Verify that this function T satisfies the boundary conditions. Verify integration formula by differentiating the right- hand side there with respect to ty.

Suggestion: The trigonometric identities

2 ty 1 + cos ty 2 ty 1 — cos ty

cos —= , sin —=

2 2 2 2

are useful in this verification.

With the aid of the trigonometric identities

tan a — tan f 2

Notes

tan (or — ft)=-, tana + coto1 =

1 + tan a tan f 'sin 2a

show how solution is obtained from the expression for nV(r, 0) just prior to that solution.

Let I denote this finite unit impulse function:

[1/h when 00 < 0 < 00 + h,

I(h,0 — 00)=,

10 when 0 < 0 < 00 or 00 + h < 0 < 2n,

where h is a positive number and 0 < 00 <00 + h < 2n. Note that

f 00+h

I(h,0 — 00) d0=1.

With the aid of a mean value theorem for definite integrals, show that

r 2n n 00+h

I P(r0,r,f — 0)I(h,f — 00) df=P(r0,r,c — 0)1 I(h,f — 00) df,

0 00

where 00 < c < 00 + h, and hence that p 2n

lim I P(r0, r,f — 0) I(h, f — 00) df=P(r0, r,0 — 00) (r < r0).

h^0 l0 h>0 J[i

Thus the Poisson kernel P(r0, r,0 — 00) is the limit, as h approaches 0 through positive values, of the harmonic function inside the circle r=r0 whose boundary values are represented by the impulse function 2nI(h,0 — 00).

Show that the expression in for the sum of a certain cosine series can be written

J _ a2 1 + 2 a" cos nO= -r (— 1 < a < 1).

1-2a cos 9 + a2 n=1

14.7 SCHWARZ INTEGRAL FORMULA

Let f be an analytic function of z throughout the half plane Im z > 0 such that for some positive constants a and M, the order property

\zaf(z)\ <M (Imz > 0)

is satisfied. For a fixed point z above the real axis, let CR denote the upper half of a positively oriented circle of radius R centered at the origin, where R > \z| Then, according to the Cauchy integral formula,

1 f f(s) ds 1 rR f(t) dt

fiz)=— J—— + — J--d—.

2ni Jcr s - z 2m J_r t - z

We find that the first of these integrals approaches 0 as R tends to ro since, in view of condition ensures that the improper integral here converges. The number to which it converges is the same as its Cauchy principal value and representation is a Cauchy integral formula for the half plane Im z > 0.

When the point z lies below the real axis, the right-hand side of equation is zero; hence integral is zero for such a point. Thus, when z is above the real axis, we have the following formula, where c is an arbitrary complex constant:

1 C/ 1 c \

f(z)=— (+ -—-)f(t)dt (Irn z > 0).

2ni J_oo \t -z t -z)

In the two cases c=— 1 and c=1, this reduces, respectively, to

ra m 1 r yf(t) m

f(z)=- -y dt (y > 0)

n J—TO \t — z|2

Notes

and

1 f ~ (t — x)f(t)

f(z)=— — dt (y > 0).

ni J—TO \t — z\2

If f(z)=u(x,y) + lv(x,y), it follows from formulas that the harmonic functions u and v are represented in the half plane y > 0 in terms of the boundary values of u by the formulas

m , . 1 /■"»«.<>),_ If yult.O)

- nL«* v ' and

, x 1 f(x — t)u(t, 0)

(S) «*•»-■; lv>0'

ormula is known as the Schwarz integral formula, or the Poisson integral formula for the half plane. In the next section, we shall relax the conditions for the validity of formulas.

14.8 DIRICHLET PROBLEM FOR A HALF PLANE

Let F denote a real-valued function of x that is bounded for all x and continuous except for at most a finite number of finite jumps. When y > e and \x\< 1/e, where e is any positive constant, the integral

00 F(t) dt

oo (t-x)2 + y2

converges uniformly with respect to x and y, as do the integrals of the partial derivatives of the integrand with respect to x and y. Each of these integrals is the sum of a finite number of improper or definite integrals over intervals where F is continuous; hence the integrand of each component integral is a continuous function of t, x, and y when y > e. Consequently, each partial derivative of I(x, y) is represented by the integral of the corresponding derivative of the integrand when- ever y > 0.

If we write

U(x,y)=-I(x,y), then U is the Schwarz integral transform of F, suggested by expression

1 p COyF(t)

m = (,»0).

Except for the factor 1/n, the kernel here is y/\t — z\2. It is the imaginary component of the function 1/(t — z), which is analytic in z when y > 0. It follows that the kernel is harmonic, and so it satisfies Laplace's equation in x and y. Because the order of differentiation and integration can be interchanged, the function then satisfies that equation. Consequently, U is harmonic when y > 0.

To prove that

lim U(x,y)=F(x)

y^0 y>0

for each fixed x at which F is continuous, we substitute t=x + y tan t in integral and write

1 rn/2

U(x, y)=— I F(x + y tanr) dr (y > 0).

n J—n/2

As a consequence, if

G(x, y, t)=F(x + y tan t) — F(x)

and a is some small positive constant,

r n/2 n [U (x, y) — F(x)]=G(x,y,T)dT=h(y) + h(y) + h(y)

J —n/2

where

(—n/2)+a r (n/2)—a

G(x, y, t) dT, I2(y) = G(x,y,T)dT,

Notes

-n/2 J (—n/2)+a f n/2

I3(y) = G(x, y, t) dT.

J (n/2)—a

If M denotes an upper bound for \F(x*)|, then \G(x, y,T)\ < 2M. For a given positive number e, we select a so that 6Ma < e; and this means that e e

|/i(y)| <2Ma<- and |/3(y)| < 2Ma <

We next show that corresponding to e, there is a positive number S such that

e\h(y)\ < j whenever 0 < y < 8.

To do this, we observe that since F is continuous at x, there is a positive number Y such that

\G(x, y, r)| < —- whenever 0 < y\ tan r| < y. 3n

Now the maximum value of | tan t | as t ranges from

n n "hcf to — — cf 2 2

(n \ tan a=cot a. \ 2)

Hence, if we write S=y tan a, it follows that

Ee |/2(v)| < —(n — 2a) < - whenever 0 < v < 5. 3n 3

We have thus shown that

IC(y)l + IL(y)l + IL(y)l <e whenever 0 < y < S.

Condition now follows from this result and equation.

Formula therefore solves the Dirichlet problem for the half plane y > 0, with the boundary condition. It is evident from the form of expression that IU(x,y)I < M in the half plane, where M is an upper bound of |F(x)|; that is,

U is bounded. We note that U(x,y)=F0 when F(x)=F0 , where F0 is a constant.

According to formula under certain conditions on F the function

1 f ~ (x - t)F(t) V(x, y)=— dt (y > 0)

n J_TO (t - x)2 + y2

is a harmonic conjugate of the function U given by formula. Actually, formula furnishes a harmonic conjugate of U if F is everywhere continuous, except for at most a finite number of finite jumps, and if F satisfies an order property

IxaF(x)I < M (a > 0).

For, under those conditions, we find that U and V satisfy the Cauchy-Riemann equations when y > 0.

Special cases of formula when F is an odd or an even function

EXERCISES

Obtain as a special case of formula the expression1

F(t) dt (x > 0,y > 0)

_ (t — x)9 + y2 (t + x)2 + y2 _

for a bounded function U that is harmonic in the first quadrant and satisfies the boundary conditions

U(0, y)=0 (y > 0), lim U(x,y)=F(x) (x > 0,x=xj),

y^0 y>0

where F is bounded for all positive x and continuous except for at most a finite number of finite jumps at the points xj (j=1, 2,..., n).

Let T(x, y) denote the bounded steady temperatures in a plate x > 0,y > 0, with insulated faces, when

lim T(x, y)=F1(x) (x > 0),

y^0 y>0 lim T(x,y)=F2(y) (y > 0) x^0 x>0

Notes

Here F1 and F2 are bounded and continuous except for at most a finite number of finite jumps. Write x + iy=z and show with the aid of the expression obtained in Exercise that

T(x,y)=T1(x,y) + T2(x,y) (x > 0,y > 0)\it — z\2 \it + z\2

(t — x)2 + y2 (t + x)2 + y2

for a bounded function U that is harmonic in the first quadrant and satisfies the boundary conditions

Ux(0,y)=0 (y > 0),

lim U(x,y)=F(x) (x > 0,x=xi). y^0 y>0

where F is bounded for all positive x and continuous except possibly for finite jumps at a finite number of points x=xj (j=1. 2.....n).

Interchange the x and y axes

1 rxF(t)

Ui'-'"=*L 0-»)' + v'" ('*>0)

of the Dirichlet problem for the half plane x > 0. Then write

F(y)- I1 when yl < 1

(y) |0 when |y| > 1. and obtain these expressions for U and its harmonic conjugate — V:

U(x, y)=- farctan — - arctan ^ Y V(x, y)=In X] + + ^ \ x x)
2n x10 + (y — 1)2

where —n/2 < arctan t < n/2. Also, show that

V(x, v) + iU(x, v)=— [Log(z + /) - Log(z - /)], n where z=x + iy.

14.9 NEUMANN PROBLEMS

we write

s=r0exp(i<^) and z=r exp(i0) (r < r0).

When s is fixed, the function

<2(r0. r. <p — 0)=—2r0 ln|s — z|=-r0ln[r02 — 2r0r cos(<£ — 0) + r2] is harmonic interior to the circle |z|=r0 because it is the real component of

—2 r0 log (z — s).

where the branch cut of log(z — s) is an outward ray from the point s. If, moreover, r=0,

These observations suggest that the function Q may be used to write an integral representation for a harmonic function U whose normal derivative Ur on the circle r=r0 assumes prescribed values G(Q).

If G is piecewise continuous and U0 is an arbitrary constant, the function

1 f2n lJ(r,6)=-— I Q(r0,r,<p-0)G(<p) d<p + U0 (r < r0) 2n J0

is harmonic because the integrand is a harmonic function of r and Q. If the mean value of G over the circle |z|=r0 is zero, so that

p 2n / G(f) d$=0, 0

then, in view of equation,

1 p2n r

Ur(r,0)=— —[P(r0,r,(p -0) -l]G((p) dip 2n J0 r r0 1 f2n = -•—/P(r$, r, <p — 0) G(<p) dtp. r 2n J0

Now, according to equations,

1 f2n lim — P(ro,r,<p-0)G(<p) d<p=G(0). r^ra 2n J0

r<rQ   0 u

Hence

lim Ur(r,Q)=G(Q)

r^r0 r<rg

for each value of Q at which G is continuous.

When G is piecewise continuous and satisfies condition (4), the formula

Notes

r C 2n U(r,0)=-^- \n[rl - 2r0rcos(</> - 0) + r2] G(cp) dip + t/0 (r < r0),

2n J0

therefore, solves the Neumann problem for the region interior to the circle r=r0 where G(Q) is the normal derivative of the harmonic function U(r,Q) at the bound- ary in the sense of condition. Note how it follows from equations and that since ln r0 is constant, U0 is the value of U at the center r=0 of the circle r=r0.

The values U(r,Q) may represent steady temperatures in a disk r < r0 with insulated faces. In that case, condition states that the flux of heat into the disk through its edge is proportional to G(Q). Condition is the natural phys- ical requirement that the total rate of flow of heat into the disk be zero, since temperatures do not vary with time.

A corresponding formula for a harmonic function H in the region exterior to the circle r=r0 can be written in terms of Q as

1 r2n H(R, f)=—— I Q(ro,R,4>-f)G(4»d4> + Ho (R > r0), 2n J0

where Ho is a constant. As before, we assume that G is piecewise continuous and that condition holds. Then

H0=lim H(R,f) R^to and

lim HR(R,f)=G(f) R^r0 R>r0

for each f at which G is continuous. Verification of formula, as well as special cases of formula that apply to semicircular regions, is left to the exercises.

Turning now to a half plane, we let G(x) be continuous for all real x, except possibly for a finite number of finite jumps, and let it satisfy an order property

\xaG(x)\ < M (a > 1)

when —to < x < to. For each fixed real number t, the function Log\z — t\is har- monic in the half plane Im z > 0. Consequently, the function

TO

U(x, y)=— I ln|z — t\ G(t) dt + t/0TO r(t — x)2 + y2

(t + x)2 + y2 (mx.y)=±f In 2n J0

This represents a function that is harmonic in the first quadrant $x > 0$, $y > 0$ and satisfies the boundary conditions

U(0,y)=0 ($y > 0$),

lim Uy(x, y)=G(x) ($x > 0$). y^0 y>0

Check your Progress-1

Discuss Schwarz-Christoffel Transformation

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Discuss Poisson Integral Formula

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14.10 LET US SUM UP

In this unit we have discussed the definition and example of Schwarz-Christoffel Transformation, Triangles And Rectangles, Integral Formulas Of The Poisson Type, Poisson Integral Formula, Dirichlet Problem For A Disk, Schwarz Integral Formula, Dirichlet Problem For A Half Plane, Neumann Problems

14.11 KEYWORDS

Schwarz-Christoffel Transformation In our expression f'(z)=A(z — X1)—k1 (z — X2)—k2 ■■■(z — Xn — 1)—kn—1

Triangles And Rectangles The Schwarz-Christoffel transformation is written in terms of the points xj and not in terms of their images, which are the vertices of the polygon

Integral Formulas Of The Poisson Type In this chapter, we develop a theory that enables us to solve a variety of boundary value problems whose solutions are expressed in terms of definite or improper integrals. Many of the integrals occurring are then readily evaluated.

Poisson Integral Formula Let C0 denote a positively oriented circle, centered at the origin, and suppose that a function f is analytic inside and on C0.

Dirichlet Problem For A Disk Let F be a piecewise continuous function of Q on the interval 0 < Q < 2n. The Poisson integral transform of F is defined in terms of the Poisson kernel P(r0, r , p — Q),

Schwarz Integral Formula Let f be an analytic function of z throughout the half plane Im z > 0 such that for some positive constants a and M, the order property \zaf(z)\ <M (Imz > 0)

Dirichlet Problem For A Half Plane Let F denote a real-valued function of x that is bounded for all x and continuous except for at most a finite number of finite jumps

Neumann Problems we write s=r0exp(i<^) and z=r exp(i0) (r < r0) When s is fixed, the function

14.12 QUESTIONS FOR REVIEW

Explain Schwarz-Christoffel Transformation

Explain Poisson Integral Formula

14.13 ANSWERS TO CHECK YOUR PROGRESS

Schwarz-Christoffel Transformation

(answer for Check your Progress-1 Q)

Poisson Integral Formula (answer for Check your Progress-1 Q)

14.14 REFERENCES

- Complex Analysis
- Basic of Complex Analysis
- Complex Functions & Variables
- Complex Variables
- Introduction To Complex Analysis
- Application Of Complex Analysis & Variables
- Complex Functions
- Complex Numbers & Analysis
- The Complex Number Syste